On the Limit Points of Certain Geometric Progressions modulo 1

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Abstract: A seminal result by Weyl states that for an irrational number a, the sequence $a, 2a, 3a, \dots$ is equidistributed modulo 1, meaning the proportion of terms $(\{na\})_n$ falling within any subinterval of $[0, 1)$ is equal to the length of that subinterval, where $\{\cdot\}$ denotes the fractional part. Many other sequences are known to be equidistributed in $\mathbb{R}/\mathbb{Z}$. For instance, geometric progressions of the form $(\xi\alpha^n)_n$ are uniformly distributed modulo 1 for a fixed $\xi > 0$ and almost every $\alpha > 1$, or conversely, for a fixed $\alpha > 1$ and almost every $\xi > 0$. When both α and ξ are fixed, the problem becomes significantly more challenging. A theorem by Pisot states that, if α is algebraic, then such a sequence has finitely many limit points if and only if α is a Pisot number and $\xi \in \mathbb{Q}(\alpha)$. A fascinating problem arising from this context is whether, given a Pisot number α, one can find a ξ such that the geometric progression modulo 1 has a prescribed number of limit points. This question is closely related to the problem of identifying linear recurrences with a fixed characteristic polynomial that exhibit a predetermined number of residues modulo an integer. The aim of this talk is to delve deeper into these problems and present some recent findings in this area.