Prime Density of Lehmer Sequences

Christian Ballot

University of Caen-Normandie, France

Given a rational prime p and an integer sequence $X = (X_n)_{n\geq 0}$, we say that p divides X if p divides at least one term X_k of X. The prime density of X, if it exists, is defined as

$$\lim_{x\to\infty} \frac{\#\{p \leq x;\ p \text{ divides } X\}}{\pi(x)}.$$

The companion Lucas sequence $V = V(P,Q) = (V_n)_{n\geq 0}$ is the second-order linear recurrence with initial values $V_0 = 2$, $V_1 = P$ and characteristic polynomial $x^2 - Px + Q$, where P, Q are two nonzero integers. Thanks to the work of Hasse in the 1960s and of Lagarias in 1985, we know the prime density of a V sequence exists and we know how to compute it.

In his 1930 doctoral thesis, D. H. Lehmer replaced P by $\sqrt{R}$, R a non-square integer, in $x^2 - Px + Q$ and was able to define a pair of integral recurrences $\bar{U}(R,Q)$, $\bar{V}(R,Q)$, now called the Lehmer sequences, that have properties analogous to those of ordinary Lucas sequences. In this talk, we examine the question of the prime density of the companion Lehmer sequences $\bar{V}(R,Q)$.