



Politecnico
di Torino

Dipartimento di Scienze
Matematiche "G. L. Lagrange"

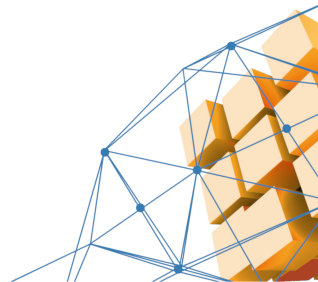


On the convergence and the periodicity of p -adic continued fractions

Giuliano Romeo

6th Number Theory Meeting

22-23 September, 2022



Classical continued fractions

Definition (Continued fraction)

A *continued fraction* is an object of the form

$$b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \ddots}}.$$

The coefficients b_n are called *partial quotients*.

Classical continued fractions

Any real number α can be represented as $[b_0, b_1, \dots]$, where the partial quotients are evaluated by

$$\begin{cases} b_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \frac{1}{\alpha_n - b_n}. \end{cases}$$

Classical continued fractions

Any real number α can be represented as $[b_0, b_1, \dots]$, where the partial quotients are evaluated by

$$\begin{cases} b_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \frac{1}{\alpha_n - b_n}. \end{cases}$$

Example

The continued fraction of $\frac{11}{4}$ in \mathbb{R} is:

$$\frac{11}{4}$$

Classical continued fractions

Any real number α can be represented as $[b_0, b_1, \dots]$, where the partial quotients are evaluated by

$$\begin{cases} b_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \frac{1}{\alpha_n - b_n}. \end{cases}$$

Example

The continued fraction of $\frac{11}{4}$ in \mathbb{R} is:

$$\frac{11}{4} = 2 + \frac{3}{4} =$$

Classical continued fractions

Any real number α can be represented as $[b_0, b_1, \dots]$, where the partial quotients are evaluated by

$$\begin{cases} b_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \frac{1}{\alpha_n - b_n}. \end{cases}$$

Example

The continued fraction of $\frac{11}{4}$ in \mathbb{R} is:

$$\frac{11}{4} = 2 + \frac{3}{4} = 2 + \frac{1}{\frac{4}{3}}$$

Classical continued fractions

Any real number α can be represented as $[b_0, b_1, \dots]$, where the partial quotients are evaluated by

$$\begin{cases} b_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \frac{1}{\alpha_n - b_n}. \end{cases}$$

Example

The continued fraction of $\frac{11}{4}$ in \mathbb{R} is:

$$\frac{11}{4} = 2 + \frac{3}{4} = 2 + \frac{1}{\frac{4}{3}} = 2 + \frac{1}{1 + \frac{1}{3}} = [2, 1, 3].$$

Classical continued fractions

Theorem

The continued fraction of $\alpha \in \mathbb{R}$ is **finite** if and only if α is a **rational**.

Theorem (Lagrange's Theorem)

The continued fraction of $\alpha \in \mathbb{R}$ is **periodic** if and only if α is a **quadratic irrational**.

Classical continued fractions

Theorem

The continued fraction of $\alpha \in \mathbb{R}$ is **finite** if and only if α is a **rational**.

Theorem (Lagrange's Theorem)

The continued fraction of $\alpha \in \mathbb{R}$ is **periodic** if and only if α is a **quadratic irrational**.

$$\sqrt{2} = [1, \overline{2}] = 1 + \frac{1}{2 + \frac{1}{2 + \ddots}}, \quad \phi = [\overline{1}] = 1 + \frac{1}{1 + \frac{1}{1 + \ddots}}.$$

The field of p -adic numbers

Definition (p -adic valuation)

Let $p \in \mathbb{Z}$ be a prime number. The **p -adic valuation** of $\alpha \in \mathbb{Q}$ is:

$$v_p(\alpha) = e, \quad \text{such that } n = p^e \frac{a}{b} \text{ and } p \nmid ab,$$

for all $\alpha \neq 0$, and $v_p(0) = +\infty$.

The field of p -adic numbers

Definition (p -adic valuation)

Let $p \in \mathbb{Z}$ be a prime number. The **p -adic valuation** of $\alpha \in \mathbb{Q}$ is:

$$v_p(\alpha) = e, \quad \text{such that } n = p^e \frac{a}{b} \text{ and } p \nmid ab,$$

for all $\alpha \neq 0$, and $v_p(0) = +\infty$.

Definition (p -adic absolute value)

Let $p \in \mathbb{Z}$ be a prime number. The **p -adic absolute value** of $\alpha \in \mathbb{Q}$ is:

$$|\alpha|_p = \frac{1}{p^{v_p(\alpha)}},$$

for all $\alpha \neq 0$, and $|0|_p = 0$.



The field of p -adic numbers

The field of p -adic numbers \mathbb{Q}_p is obtained as the completion of \mathbb{Q} with respect to the p -adic absolute value.

The field of p -adic numbers

The field of p -adic numbers \mathbb{Q}_p is obtained as the completion of \mathbb{Q} with respect to the p -adic absolute value.

\mathbb{Q}_p is the set of all finite-tailed formal power series in p :

$$\mathbb{Q}_p = \left\{ \sum_{n=-r}^{+\infty} a_n p^n \mid r \in \mathbb{Z}, a_n \in \mathbb{Z}/p\mathbb{Z} \right\}.$$

Continued fractions in the field of p -adic numbers

The main goal of the research in p -adic continued fractions is to find an algorithm that, given $\alpha \in \mathbb{Q}_p$, produces a continued fraction that:

Continued fractions in the field of p -adic numbers

The main goal of the research in p -adic continued fractions is to find an algorithm that, given $\alpha \in \mathbb{Q}_p$, produces a continued fraction that:

- a) converges p -adically to α ,

Continued fractions in the field of p -adic numbers

The main goal of the research in p -adic continued fractions is to find an algorithm that, given $\alpha \in \mathbb{Q}_p$, produces a continued fraction that:

- a) converges p -adically to α ,
- b) is finite if and only if α is rational,

Continued fractions in the field of p -adic numbers

The main goal of the research in p -adic continued fractions is to find an algorithm that, given $\alpha \in \mathbb{Q}_p$, produces a continued fraction that:

- a) converges p -adically to α ,
- b) is finite if and only if α is rational,
- c) is periodic if and only if α is a quadratic irrational.

Browkin's first algorithm

Let $\alpha = \sum_{n=-r}^{+\infty} a_n p^n \in \mathbb{Q}_p$. Browkin defined the first floor function as

$$s(\alpha) = \sum_{n=-r}^0 a_n p^n \in \mathbb{Q}.$$

Browkin's first algorithm

Let $\alpha = \sum_{n=-r}^{+\infty} a_n p^n \in \mathbb{Q}_p$. Browkin defined the first floor function as

$$s(\alpha) = \sum_{n=-r}^0 a_n p^n \in \mathbb{Q}.$$

Given $\alpha_0 \in \mathbb{Q}_p$, his first algorithm works as follows:

$$\begin{cases} b_n = s(\alpha_n) \\ \alpha_{n+1} = \frac{1}{\alpha_n - s(\alpha_n)}. \end{cases}$$

Browkin's second algorithm

Let $\alpha = \sum_{n=-r}^{+\infty} a_n p^n \in \mathbb{Q}_p$. Browkin defined the second floor function as

$$t(\alpha) = \sum_{n=-r}^{-1} a_n p^n \in \mathbb{Q}.$$

Browkin's second algorithm

Let $\alpha = \sum_{n=-r}^{+\infty} a_n p^n \in \mathbb{Q}_p$. Browkin defined the second floor function as

$$t(\alpha) = \sum_{n=-r}^{-1} a_n p^n \in \mathbb{Q}.$$

Given $\alpha_0 \in \mathbb{Q}_p$, his second algorithm works as follows:

$$\begin{cases} b_n = s(\alpha_n) & \text{if } n \text{ even} \\ b_n = t(\alpha_n) & \text{if } n \text{ odd and } v_p(\alpha_n - t(\alpha_n)) \neq 0 \\ b_n = t(\alpha_n) - \text{sign}(t(\alpha_n)) & \text{if } n \text{ odd and } v_p(\alpha_n - t(\alpha_n)) = 0 \\ \alpha_{n+1} = \frac{1}{\alpha_n - s(\alpha_n)}. \end{cases}$$

Browkin's continued fractions

Lemma (J. Browkin, 1978)

Let $b_0, b_1, \dots \in \mathbb{Z}[\frac{1}{p}]$ be such that, for all $n \in \mathbb{N}$,

$$v_p(b_n) < 0.$$

Then the continued fraction $[b_0, b_1, \dots]$ converges to an element $\alpha \in \mathbb{Q}_p$.

Lemma (J. Browkin, 2000)

Let $b_0, b_1, \dots \in \mathbb{Z}[\frac{1}{p}]$ be such that, for all $n \in \mathbb{N}$,

$$\begin{cases} v_p(b_{2n+1}) < 0 \\ v_p(b_{2n+2}) = 0. \end{cases}$$

Then the continued fraction $[b_0, b_1, \dots]$ converges to an element $\alpha \in \mathbb{Q}_p$.

Browkin's continued fractions



Theorem (J. Browkin, 1978)

Every rational number has a finite continued fraction expansion with Browkin I.

Theorem (S. Barbero, U. Cerruti, N. Murru, 2021)

Every rational number has a finite continued fraction expansion with Browkin II.

Some results

-  N. Murru, G. Romeo, G. Santilli, *Convergence conditions for p -adic continued fractions*,
preprint (2022), available at: <https://arxiv.org/abs/2202.09249>.
Submitted to *Research in Number Theory*.
-  N. Murru, G. Romeo, G. Santilli, *Periodicity of an algorithm for p -adic continued fractions*,
preprint (2022), available at: <https://arxiv.org/abs/2201.12019>.
Submitted to *Annali di Matematica Pura e Applicata*.



Convergence conditions for p -adic continued fractions

Problem

Is it possible to generalize it in 3 or more steps?

Convergence conditions for p -adic continued fractions

Problem

Is it possible to generalize it in 3 or more steps? For example, for all $n \in \mathbb{N}$,

$$\begin{cases} v_p(b_{3n+1}) < 0 \\ v_p(b_{3n+2}) = 0 \\ v_p(b_{3n+3}) = 0. \end{cases}$$

Convergence conditions for p -adic continued fractions

Problem

Is it possible to generalize it in 3 or more steps? For example, for all $n \in \mathbb{N}$,

$$\begin{cases} v_p(b_{3n+1}) < 0 \\ v_p(b_{3n+2}) = 0 \\ v_p(b_{3n+3}) = 0. \end{cases}$$



Convergence conditions for p -adic continued fractions

Theorem

Let $b_0, b_1, \dots \in \mathbb{Z}[\frac{1}{p}]$ be such that, for all $n \in \mathbb{N}$,

$$\begin{cases} v_p(b_{3n+1}) < 0 \\ v_p(b_{3n+2}) = 0 \\ v_p(b_{3n+3}) = 0. \end{cases}$$

If

$$v_p(b_{3n+2}b_{3n+3} + 1) = 0 \quad \text{for all } n \in \mathbb{N},$$

then the continued fraction $[b_0, b_1, \dots]$ converges in \mathbb{Q}_p .

A new algorithm

Let $\alpha = \sum_{n=-r}^{+\infty} a_n p^n \in \mathbb{Q}_p$. In combination with the two functions of Browkin,

$$s(a) = \sum_{n=-r}^0 a_n p^n, \quad t(a) = \sum_{n=-r}^{-1} a_n p^n,$$

we define the third floor function as:

$$u(\alpha) = \begin{cases} +1 & \text{if } a_0 \in \left\{ +2, \dots, \frac{p-1}{2} \right\} \cup \{-1\} \\ -1 & \text{if } a_0 \in \left\{ -\frac{p-1}{2}, \dots, -2 \right\} \cup \{+1\}. \end{cases}$$

A new algorithm

On input $\alpha_0 = \alpha$, for $n \geq 0$, the new algorithm works as follows:

$$\left\{ \begin{array}{ll} b_n = s(\alpha_n) & \text{if } n \equiv 0 \pmod{3} \\ b_n = t(\alpha_n) & \text{if } n \equiv 1 \pmod{3} \text{ and } v_p(\alpha_n - t(\alpha_n)) = 0 \\ b_n = t(\alpha_n) - \text{sign}(t(\alpha_n)) & \text{if } n \equiv 1 \pmod{3} \text{ and } v_p(\alpha_n - t(\alpha_n)) \neq 0 \\ b_n = s(\alpha_n) - u(\alpha_n) & \text{if } n \equiv 2 \pmod{3} \\ \alpha_{n+1} = \frac{1}{\alpha_n - b_n}. \end{array} \right.$$

A new algorithm

On input $\alpha_0 = \alpha$, for $n \geq 0$, the new algorithm works as follows:

$$\left\{ \begin{array}{ll} b_n = s(\alpha_n) & \text{if } n \equiv 0 \pmod{3} \\ b_n = t(\alpha_n) & \text{if } n \equiv 1 \pmod{3} \text{ and } v_p(\alpha_n - t(\alpha_n)) = 0 \\ b_n = t(\alpha_n) - \text{sign}(t(\alpha_n)) & \text{if } n \equiv 1 \pmod{3} \text{ and } v_p(\alpha_n - t(\alpha_n)) \neq 0 \\ b_n = s(\alpha_n) - u(\alpha_n) & \text{if } n \equiv 2 \pmod{3} \\ \alpha_{n+1} = \frac{1}{\alpha_n - b_n}. \end{array} \right.$$

Theorem

For $\alpha \in \mathbb{Q}$ the new algorithm stops in a finite number of steps.

Periodicity of p -adic continued fractions

Theorem (E. Bedocchi, 1988)

Let $\alpha \in \mathbb{Q}_p$ have a periodic continued fraction expansion with Browkin I. Then the expansion is purely periodic, i.e. $\alpha = [\overline{b_0, \dots, b_{k-1}}]$, if and only if

$$|\alpha|_p > 1, \quad |\bar{\alpha}|_p < 1.$$

Periodicity of p -adic continued fractions

Theorem (E. Bedocchi, 1988)

Let $\alpha \in \mathbb{Q}_p$ have a periodic continued fraction expansion with Browkin I. Then the expansion is purely periodic, i.e. $\alpha = [\overline{b_0, \dots, b_{k-1}}]$, if and only if

$$|\alpha|_p > 1, \quad |\bar{\alpha}|_p < 1.$$

Theorem (E. Bedocchi, 1989)

The **preperiod** for square roots of integers having periodic continued fraction expansion with Browkin I can only have length 2 or 3.

Periodicity of p -adic continued fractions

Theorem (E. Bedocchi, 1988)

Let $\alpha \in \mathbb{Q}_p$ have a periodic continued fraction expansion with Browkin I. Then the expansion is purely periodic, i.e. $\alpha = [\overline{b_0, \dots, b_{k-1}}]$, if and only if

$$|\alpha|_p > 1, \quad |\bar{\alpha}|_p < 1.$$

Theorem (E. Bedocchi, 1989)

The **preperiod** for square roots of integers having periodic continued fraction expansion with Browkin I can only have length 2 or 3.

There exist infinitely many square roots of integers with periodic Browkin I expansion of **period** length 2.

Periodicity of p -adic continued fractions

Theorem

If $\alpha \in \mathbb{Q}_p$ has a purely periodic Browkin II expansion

$$\alpha = [\overline{b_0, \dots, b_{k-1}}],$$

then

$$|\alpha|_p = 1, \quad |\overline{\alpha}|_p < 1.$$



Periodicity of p -adic continued fractions

The viceversa is not true in general.

Periodicity of p -adic continued fractions

The viceversa is not true in general. In fact, if we consider

$$\sqrt{30} = 3 - 3 \cdot 7 + \dots \in \mathbb{Q}_7,$$

then

$$\sqrt{30} + 3 = \left[-1, \frac{3}{7}, 3, \frac{2}{7}, 1, \frac{2}{7}, -2, \frac{3}{7}, 1, \frac{2}{7}, 2, \frac{1}{7}, -1, -\frac{5}{7} \right],$$

Periodicity of p -adic continued fractions

The viceversa is not true in general. In fact, if we consider

$$\sqrt{30} = 3 - 3 \cdot 7 + \dots \in \mathbb{Q}_7,$$

then

$$\sqrt{30} + 3 = \left[-1, \frac{3}{7}, 3, \frac{2}{7}, 1, \frac{2}{7}, -2, \frac{3}{7}, 1, \frac{2}{7}, 2, \frac{1}{7}, -1, -\frac{5}{7} \right],$$

although the hypothesis of the previous theorem are satisfied:

$$\begin{aligned} |3 + \sqrt{30}|_7 &= 1, \\ |3 - \sqrt{30}|_7 &= |3 \cdot 7 + \dots| < 1. \end{aligned}$$

Periodicity of p -adic continued fractions

Theorem

Let $\alpha \in \mathbb{Q}_p$ with periodic Browkin II expansion

$$\alpha = [b_0, b_1, \dots, b_{h-1}, \overline{b_h, \dots, b_{h+k-1}}].$$

Then, if

$$|\alpha|_p = 1, \quad |\bar{\alpha}|_p < 1,$$

the preperiod length h is even.

Periodicity of p -adic continued fractions

Theorem

Let $\alpha \in \mathbb{Q}_p$ with periodic Browkin II expansion

$$\alpha = [b_0, b_1, \dots, b_{h-1}, \overline{b_h, \dots, b_{h+k-1}}].$$

Then, if

$$|\alpha|_p = 1, \quad |\bar{\alpha}|_p < 1,$$

the preperiod length h is even.

Corollary

Let \sqrt{D} be defined in \mathbb{Q}_p , with $D \in \mathbb{Z}$ not a square; then, if \sqrt{D} has a periodic continued fraction with Browkin II, the preperiod has length either 1 or even.



Periodicity of p -adic continued fractions

For every p prime, Browkin proved that there exist infinitely many square roots of integers having periods of length 2, leaving open the same question for the periods of length 4.

Periodicity of p -adic continued fractions

For every p prime, Browkin proved that there exist infinitely many square roots of integers having periods of length 2, leaving open the same question for the periods of length 4.

Theorem

For any integer $t \geq 2$, given $D = \frac{1 - p^t}{(1 - p)^2} \cdot p^2$, then

$$\pm\sqrt{D} = \left[0, \pm\frac{1}{p}, \overline{\mp 1, \mp \frac{2(p^{t-1} - 1)}{(p - 1)p^{t-1}}, \mp 1, \pm\frac{2}{p}} \right],$$

and there are infinitely many D that are integers.



Conclusions and further research

In future works it is worth to:

Conclusions and further research

In future works it is worth to:






- Deepen the study of the **periodicity properties** of Browkin's algorithms, in particular *Browkin II*, in order to approach a characterization for periodic p -adic continued fractions,

Conclusions and further research

In future works it is worth to:

- Deepen the study of the **periodicity properties** of Browkin's algorithms, in particular *Browkin II*, in order to approach a characterization for periodic p -adic continued fractions,
- Define **more algorithms** that fulfill the convergence conditions, in order to verify if they present better periodicity properties than the existent algorithms.

Bibliography

-  J. Browkin, *Continued fractions in local fields, I*, Demonstratio Mathematica, **11** (1978), 67-82.
-  J. Browkin, *Continued fractions in local fields, II*, Mathematics of Computations, **70** (2000), 1281-1292.
-  E. Bedocchi, *Nota sulle frazioni continue p -adiche*, Ann. Mat. Pura Appl., **152** (1988), 197-207.
-  S. Barbero, U. Cerruti, N. Murru, *Periodic representations for quadratic irrationals in the field of p -adic numbers*, Mathematics of Computation, **90** (2021), 2267-2280.
-  L. Capuano, N. Murru, L. Terracini, *On periodicity of p -adic Browkin continued fractions*, preprint, (2020), available at: <https://arxiv.org/abs/2010.07364>.



Grazie per l'attenzione



Politecnico
di Torino