A curiosity about the sum $(-1)^{[e]} + (-1)^{[2e]} + \cdots + (-1)^{[Ne]}$

Francesco Amoroso

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen
France

Joint work with M. Omarjee (Lycée Henry IV, Paris)

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$?

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

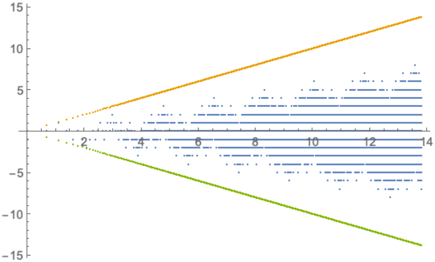
What is the asymptotic behaviour of $S_N(\alpha)$? Numerical examples:

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$? Numerical examples:



$\alpha = \sqrt{2}$, $1 \le N \le 10^6$

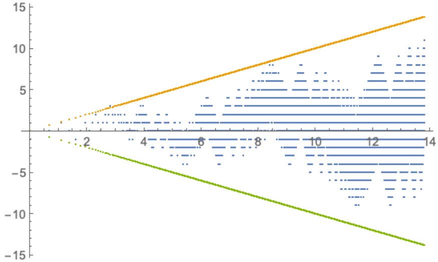
Logarithmic scale, the orange line is $\log N$ and the green one is $-\log N$.

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$? Numerical examples:



$\alpha = 2e$, $1 \leq N \leq 10^6$

Logarithmic scale, the orange line is $\log N$ and the green one is $-\log N$.

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$?

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$?

These pictures could suggest: $S_N(\log 2) = O(\log N)$

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$?

These pictures could suggest: $S_N(\log 2) = O(\log N)$ (true) and $S_N(2e) = O(\log N)$ (false).

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$?

These pictures could suggest: $S_N(\log 2) = O(\log N)$
and $S_N(2e) =$ ~~$O(\log N)$~~ $O((\log N)^2)$.

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$?

These pictures could suggest: $S_N(\log 2) = O(\log N)$
and $S_N(2e) =$ ~~$O(\log N)$~~ $O((\log N)^2)$. And for $S_N(e)$?

Motivation

Given an irrational α and a natural number N, let

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

What is the asymptotic behaviour of $S_N(\alpha)$?

These pictures could suggest: $S_N(\log 2) = O(\log N)$
and $S_N(2e) =$ ~~$O(\log N)$~~ $O((\log N)^2)$. And for $S_N(e)$?

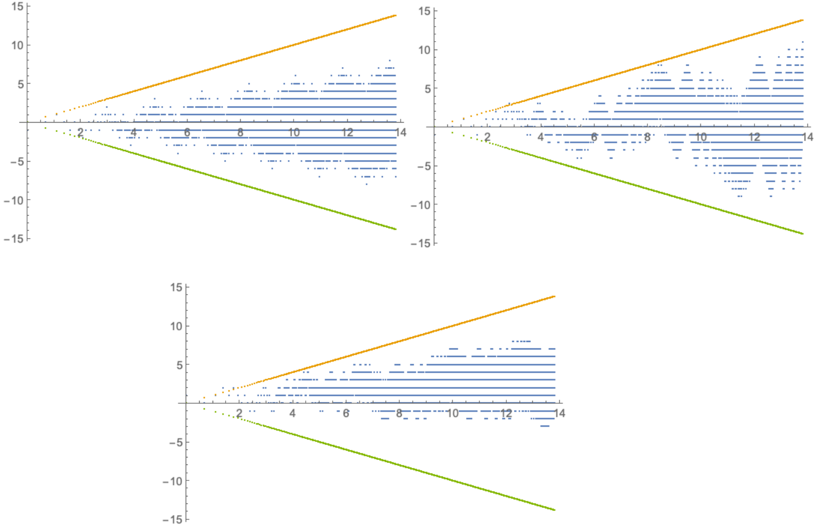
Question (Omarjee)

It is true that

$$S_N(e) = (-1)^{[e]} + (-1)^{[2e]} + \cdots + (-1)^{[ne]} = O(\log N).$$

as some numerical evidence suggests?

$S_N(\sqrt{2})$, $S_N(2e)$, $S_N(e)$



Results

The behaviour of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$$

depends on the continued fraction (c.f.) expansion of $\alpha/2$.

Results

The behaviour of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$$

depends on the continued fraction (c.f.) expansion of $\alpha/2$.

Since the c.f. of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O(\log(N))$ and this bound is best possible.

Results

The behaviour of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$$

depends on the continued fraction (c.f.) expansion of $\alpha/2$.

Since the c.f. of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O(\log(N))$ and this bound is best possible.

The partial quotients of e grow slowly. This implies:

Results

The behaviour of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$$

depends on the continued fraction (c.f.) expansion of $\alpha/2$.

Since the c.f. of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O(\log(N))$ and this bound is best possible.

The partial quotients of e grow slowly. This implies: $S_N(2e) = O\Big(\big(\frac{\log N}{\log\log N}\big)^2\Big)$, again best possible.

Results

The behaviour of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$$

depends on the continued fraction (c.f.) expansion of $\alpha/2$.

Since the c.f. of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O(\log(N))$ and this bound is best possible.

The partial quotients of e grow slowly. This implies: $S_N(2e) = O\Big(\big(\frac{\log N}{\log\log N}\big)^2\Big)$, again best possible.

The partial quotients of $e/2$ behave similarly as those of e.

Results

The behaviour of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$$

depends on the continued fraction (c.f.) expansion of $\alpha/2$.

Since the c.f. of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O(\log(N))$ and this bound is best possible.

The partial quotients of e grow slowly. This implies: $S_N(2e) = O\Big(\big(\frac{\log N}{\log\log N}\big)^2\Big)$, again best possible.

The partial quotients of $e/2$ behave similarly as those of e. Thus we do not expect a better result for $S_N(e)$.

Results

The behaviour of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$$

depends on the continued fraction (c.f.) expansion of $\alpha/2$.

Since the c.f. of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O(\log(N))$ and this bound is best possible.

The partial quotients of e grow slowly. This implies: $S_N(2e) = O\Big(\big(\frac{\log N}{\log\log N}\big)^2\Big)$, again best possible.

The partial quotients of $e/2$ behave similarly as those of e. Thus we do not expect a better result for $S_N(e)$.

Surprising enough: $S_N(e) = O\big(\frac{\log N}{\log\log N}\big)$.

Uniform distribution mod 1 and discrepancy

A sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ is uniformly distributed (u.d.) mod 1 if for every reals a, b with $0 \leq a < b \leq 1$ we have

$$\lim_{N\to+\infty} \frac{|\{1 \leq n \leq N \mid u_n \in [a, b)\}|}{N} = b - a.$$

Uniform distribution mod 1 and discrepancy

A sequence $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ is uniformly distributed (u.d.) mod 1 if for every reals a, b with $0 \leq a < b \leq 1$ we have

$$\lim_{N \to +\infty} \frac{|\{1 \leq n \leq N \mid u_n \in [a, b)\}|}{N} = b - a.$$

For $x \in (1, 1]$ and $N \in \mathbb{N}$, define the local discrepancy of $\mathbf{u}$ by:

$$D_N(\mathbf{u}, x) = |\{1 \leq n \leq N \mid u_n \in [0, x)\}| - Nx$$

Uniform distribution mod 1 and discrepancy

A sequence $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ is uniformly distributed (u.d.) mod 1 if for every reals a, b with $0 \leq a < b \leq 1$ we have

$$\lim_{N \to +\infty} \frac{|\{1 \leq n \leq N \mid u_n \in [a, b)\}|}{N} = b - a.$$

For $x \in (1, 1]$ and $N \in \mathbb{N}$, define the local discrepancy of $\mathbf{u}$ by:

$$D_N(\mathbf{u}, x) = |\{1 \leq n \leq N \mid u_n \in [0, x)\}| - Nx$$

and the global discrepancy:

$$D_N(\mathbf{u}) := \sup_{0 \leq x < y \leq 1} \Big| |\{1 \leq n \leq N \mid u_n \in [x, y)\}| - N(y - x) \Big|.$$

Uniform distribution mod 1 and discrepancy

A sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ is uniformly distributed (u.d.) mod 1 if for every reals a, b with $0 \leq a < b \leq 1$ we have

$$\lim_{N\to+\infty} \frac{|\{1 \leq n \leq N \mid u_n \in [a,b)\}|}{N} = b - a.$$

For $x \in (1,1]$ and $N \in \mathbb{N}$, define the local discrepancy of $\mathbf{u}$ by:

$$D_N(\mathbf{u}, x) = |\{1 \leq n \leq N \mid u_n \in [0,x)\}| - Nx$$

and the global discrepancy:

$$D_N(\mathbf{u}) := \sup_{0\leq x<y\leq 1} \Big| |\{1 \leq n \leq N \mid u_n \in [x,y)\}| - N(y-x) \Big|.$$

Thus $|D_N(\mathbf{u}, x)| \leq D_N(\mathbf{u})$ and $D_N(\mathbf{u}) = o(N)$ iff $\mathbf{u}$ is u.d.

$S_N(\alpha)$ and u.d.

Now consider the sequence $(n\alpha/2)$. Then

$$\begin{aligned} S_N(\alpha) &= (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]} \\ &= |\{n = 1, \ldots, N \mid [n\alpha] \text{ even } \}| - |\{n = 1, \ldots, N \mid [n\alpha] \text{ odd } \}| \\ &= 2|\{n = 1, \ldots, N \mid \{n\alpha/2\} \in [0, 1/2)\}| - N \\ &= 2D_N(\alpha/2, 1/2). \end{aligned}$$

where we write for short $D_N(\alpha/2, 1/2)$ instead of $D_N((n\alpha/2), 1/2)$.

$S_N(\alpha)$ and u.d.

Now consider the sequence $(n\alpha/2)$. Then

$$\begin{aligned} S_N(\alpha) &= (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]} \\ &= |\{ n = 1, \ldots, N \mid [n\alpha] \text{ even } \}| - |\{ n = 1, \ldots, N \mid [n\alpha] \text{ odd } \}| \\ &= 2|\{ n = 1, \ldots, N \mid \{n\alpha/2\} \in [0, 1/2) \}| - N \\ &= 2D_N(\alpha/2, 1/2). \end{aligned}$$

where we write for short $D_N(\alpha/2, 1/2)$ instead of $D_N((n\alpha/2), 1/2)$.

For an irrational α, the sequence $(n\alpha)$ is u.d. mod 1 (Bohl, Sierpiński and Weyl independently, 1909-10).

$S_N(\alpha)$ and u.d.

Now consider the sequence $(n\alpha/2)$. Then

$$\begin{aligned} S_N(\alpha) &= (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]} \\ &= |\{n = 1, \ldots, N \mid [n\alpha] \text{ even }\}| - |\{n = 1, \ldots, N \mid [n\alpha] \text{ odd }\}| \\ &= 2|\{n = 1, \ldots, N \mid \{n\alpha/2\} \in [0, 1/2)\}| - N \\ &= 2D_N(\alpha/2, 1/2). \end{aligned}$$

where we write for short $D_N(\alpha/2, 1/2)$ instead of $D_N((n\alpha/2), 1/2)$.

For an irrational α, the sequence $(n\alpha)$ is u.d. mod 1 (Bohl, Sierpiński and Weyl independently, 1909-10).

Thus, for α irrational,

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

Irrationality measures

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

More precise estimates for D_N depend on the irrationality exponent $\mu(\alpha)$ of $\alpha \in \mathbb{R}\backslash\mathbb{Q}$:

$$\inf\{\mu \in \mathbb{R} \mid \forall\varepsilon > 0,\ \exists C_\varepsilon > 0 \text{ s.t. } \forall p, q \in \mathbb{Z} \text{ with } q > 0, \quad \left|\alpha - \frac{p}{q}\right| > \frac{C_\varepsilon}{q^{\mu+\varepsilon}}.$$

Irrationality measures

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

More precise estimates for D_N depend on the irrationality exponent $\mu(\alpha)$ of $\alpha \in \mathbb{R}\backslash\mathbb{Q}$:

$$\inf\{\mu \in \mathbb{R} \mid \forall \varepsilon > 0,\ \exists C_\varepsilon > 0 \text{ s.t. } \forall p, q \in \mathbb{Z} \text{ with } q > 0,\quad \left|\alpha - \frac{p}{q}\right| > \frac{C_\varepsilon}{q^{\mu+\varepsilon}}.$$

It is well known that $\mu(\alpha) \geq 2$ with equality for almost all α.

Irrationality measures

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

More precise estimates for D_N depend on the irrationality exponent $\mu(\alpha)$ of $\alpha \in \mathbb{R}\backslash\mathbb{Q}$:

$$\inf\{\mu \in \mathbb{R} \mid \forall\varepsilon > 0,\ \exists C_\varepsilon > 0 \text{ s.t. } \forall p, q \in \mathbb{Z} \text{ with } q > 0, \quad \left|\alpha - \frac{p}{q}\right| > \frac{C_\varepsilon}{q^{\mu+\varepsilon}}.$$

It is well known that $\mu(\alpha) \geq 2$ with equality for almost all α.

We have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 1 - \frac{1}{\mu(\alpha)-1}$ (Hecke and Ostrowski independently, 1922).

Irrationality measures

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

More precise estimates for D_N depend on the irrationality exponent $\mu(\alpha)$ of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$:

$$\inf\{\mu \in \mathbb{R} \mid \forall \varepsilon > 0,\ \exists C_\varepsilon > 0 \text{ s.t. } \forall p, q \in \mathbb{Z} \text{ with } q > 0,\quad \left|\alpha - \frac{p}{q}\right| > \frac{C_\varepsilon}{q^{\mu+\varepsilon}}.$$

It is well known that $\mu(\alpha) \geq 2$ with equality for almost all α.

We have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 1 - \frac{1}{\mu(\alpha)-1}$ (Hecke and Ostrowski independently, 1922).

In particular, if $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

Continued fractions

Consider the continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, a_3, \dots]$$

The a_i are called partial quotients.

Continued fractions

Consider the continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, a_3, \dots]$$

The a_i are called partial quotients.

The growth of the partial quotients is closely related to the irrationality exponent.

Continued fractions

Consider the continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, a_3, \dots]$$

The a_i are called partial quotients.

The growth of the partial quotients is closely related to the irrationality exponent.

In particular, $\mu(\alpha) = 2$ if the partial quotients are uniformly bounded (as for quadratic irrationals).

Continued fractions

Consider the continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, a_3, \dots]$$

The a_i are called partial quotients.

The growth of the partial quotients is closely related to the irrationality exponent.

In particular, $\mu(\alpha) = 2$ if the partial quotients are uniformly bounded (as for quadratic irrationals).

But also for $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \dots]$ and more generally if the partial quotients growth at most polynomially.

Small partial quotients

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

If $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

Small partial quotients

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \le |D_N(\alpha/2)| = o(N).$$

If $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients.

Small partial quotients

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

If $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients.

In this case we have: $D_N(\alpha) = O(\log N)$ (Niederreiter, 1973).

Small partial quotients

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

If $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients.

In this case we have: $D_N(\alpha) = O(\log N)$ (Niederreiter, 1973).

This cannot be improved: for every infinite sequence $\mathbf{u}$ we have $|D_N(\mathbf{u})| \gg \log N$ (Schmidt, 1972).

Small partial quotients

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

If $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients.

In this case we have: $D_N(\alpha) = O(\log N)$ (Niederreiter, 1973).

This cannot be improved: for every infinite sequence $\mathbf{u}$ we have $|D_N(\mathbf{u})| \gg \log N$ (Schmidt, 1972).

Niederreiter result was generalised by Baxa and Schoissengeier (1994).

Small partial quotients

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

If $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients.

In this case we have: $D_N(\alpha) = O(\log N)$ (Niederreiter, 1973).

Niederreiter result was generalised by Baxa and Schoissengeier (1994). Applying their result to the continued fractions

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \dots]$$
$$e/2 = [1; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, \dots]$$

Small partial quotients

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

If $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients.

In this case we have: $D_N(\alpha) = O(\log N)$ (Niederreiter, 1973).

Niederreiter result was generalised by Baxa and Schoissengeier (1994). Applying their result to the continued fractions

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$$
$$e/2 = [1; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, \ldots]$$

both with partial quotients $a_j = O(j)$,

Small partial quotients

$$|S_N(\alpha)| = 2|D_N(\alpha/2, 1/2)| \leq |D_N(\alpha/2)| = o(N).$$

If $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$.

A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients.

In this case we have: $D_N(\alpha) = O(\log N)$ (Niederreiter, 1973).

Niederreiter result was generalised by Baxa and Schoissengeier (1994). Applying their result to the continued fractions

$$\begin{aligned} e &= [2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots] \\ e/2 &= [1; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, \ldots] \end{aligned}$$

both with partial quotients $a_j = O(j)$, we see that $D_N(e)$, $D_N(e/2) \in O\Big(\big(\frac{\log N}{\log\log N}\big)^2\Big)$, and both estimates are sharp.

Global versus local discrepancy

So far, we have motivated:

$$|S_N(e)| = 2|D_N(e/2, 1/2)| \leq |D_N(e/2)| = O\Big(\Big(\frac{\log N}{\log\log N}\Big)^2\Big).$$

and the $O()$ estimate is sharp.

Global versus local discrepancy

So far, we have motivated:

$$|S_N(e)| = 2|D_N(e/2, 1/2)| \leq |D_N(e/2)| = O\Big(\Big(\frac{\log N}{\log\log N}\Big)^2\Big).$$

and the $O()$ estimate is sharp.

Surprising enough, $|D_N(e/2, 1/2)| \leq |D_N(e/2)|$ can be improved.

Global versus local discrepancy

So far, we have motivated:

$$|S_N(e)| = 2|D_N(e/2, 1/2)| \leq |D_N(e/2)| = O\Big(\Big(\frac{\log N}{\log\log N}\Big)^2\Big).$$

and the $O()$ estimate is sharp.

Surprising enough, $|D_N(e/2, 1/2)| \leq |D_N(e/2)|$ can be improved.

Applying a recent result (2011) of Roçadas and J. Schoissengeier, we see that

$$S_N(e) = 2D_N(e/2, 1/2) = O\Big(\frac{\log N}{\log\log N}\Big).$$

Global versus local discrepancy

So far, we have motivated:

$$|S_N(e)| = 2|D_N(e/2,1/2)| \leq |D_N(e/2)| = O\Big(\Big(\frac{\log N}{\log\log N}\Big)^2\Big).$$

and the $O()$ estimate is sharp.

Surprising enough, $|D_N(e/2,1/2)| \leq |D_N(e/2)|$ can be improved.

Applying a recent result (2011) of Roçadas and J. Schoissengeier, we see that

$$S_N(e) = 2D_N(e/2,1/2) = O\Big(\frac{\log N}{\log\log N}\Big).$$

More precisely,

$$\limsup_{N\to+\infty} S_N(e)/(\tfrac{3}{4}\tfrac{\log N}{\log\log N}) = 1,$$
$$\liminf_{N\to+\infty} S_N(e)/(\tfrac{3}{12}\tfrac{\log N}{\log\log N}) = -1.$$

Why $S_N(e)$ behaves differently from $S_N(2e)$?

Recall that $S_N(e) = 2D_N(e/2, 1/2)$ and $S_N(2e) = 2D_N(e, 1/2)$

Why $S_N(e)$ behaves differently from $S_N(2e)$?

Recall that $S_N(e) = 2D_N(e/2, 1/2)$ and $S_N(2e) = 2D_N(e, 1/2)$

Let $\alpha = [a_0; a_1, a_2, a_3, \ldots]$. The global discrepancy of $(n\alpha)$ depends only on the rate of growth of a_n.

Why $S_N(e)$ behaves differently from $S_N(2e)$?

Recall that $S_N(e) = 2D_N(e/2, 1/2)$ and $S_N(2e) = 2D_N(e, 1/2)$

Let $\alpha = [a_0; a_1, a_2, a_3, \ldots]$. The global discrepancy of $(n\alpha)$ depends only on the rate of growth of a_n.

The asymptotic of the local discrepancy $D_N(\alpha, 1/2)$ depends on the rate of growth and on congruences satisfied by the partial coefficients.

Why $S_N(e)$ behaves differently from $S_N(2e)$?

Recall that $S_N(e) = 2D_N(e/2, 1/2)$ and $S_N(2e) = 2D_N(e, 1/2)$

Let $\alpha = [a_0; a_1, a_2, a_3, \ldots]$. The global discrepancy of $(n\alpha)$ depends only on the rate of growth of a_n.

The asymptotic of the local discrepancy $D_N(\alpha, 1/2)$ depends on the rate of growth and on congruences satisfied by the partial coefficients.

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Why $S_N(e)$ behaves differently from $S_N(2e)$?

Recall that $S_N(e) = 2D_N(e/2, 1/2)$ and $S_N(2e) = 2D_N(e, 1/2)$

Let $\alpha = [a_0; a_1, a_2, a_3, \ldots]$. The global discrepancy of $(n\alpha)$ depends only on the rate of growth of a_n.

The asymptotic of the local discrepancy $D_N(\alpha, 1/2)$ depends on the rate of growth and on congruences satisfied by the partial coefficients.

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Asymptotic upper bounds for $D_N(\alpha, 1/2)$ depends on $S_m = \sum_{j=1}^{m} a'_j$.

Why $S_N(e)$ behaves differently from $S_N(2e)$?

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Asymptotic upper bounds for $D_N(\alpha, 1/2)$ depends on $S_m = \sum_{j=1}^{m} a'_j$.

Why $S_N(e)$ behaves differently from $S_N(2e)$?

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Asymptotic upper bounds for $D_N(\alpha, 1/2)$ depends on $S_m = \sum_{j=1}^{m} a'_j$.

For $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ we have $\{a'_j\}_{j \geq 1} = \{1, 0, 1, 0, 4, 0, 1, 0, 1, 0, 8, \ldots\}$

Why $S_N(e)$ behaves differently from $S_N(2e)$?

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Asymptotic upper bounds for $D_N(\alpha, 1/2)$ depends on $S_m = \sum_{j=1}^m a'_j$.

For $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ we have $\{a'_j\}_{j \geq 1} = \{1, 0, 1, 0, 4, 0, 1, 0, 1, 0, 8, \ldots\}$

($a'_j = 0$ when j is even). Thus $S_m = O(m^2)$.

Why $S_N(e)$ behaves differently from $S_N(2e)$?

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Asymptotic upper bounds for $D_N(\alpha, 1/2)$ depends on $S_m = \sum_{j=1}^m a'_j$.

For $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ we have $\{a'_j\}_{j\geq 1} = \{1, 0, 1, 0, 4, 0, 1, 0, 1, 0, 8, \ldots\}$

($a'_j = 0$ when j is even). Thus $S_m = O(m^2)$.

Things go differently for $e/2$:

Why $S_N(e)$ behaves differently from $S_N(2e)$?

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Asymptotic upper bounds for $D_N(\alpha, 1/2)$ depends on $S_m = \sum_{j=1}^{m} a'_j$.

For $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ we have $\{a'_j\}_{j \geq 1} = \{1, 0, 1, 0, 4, 0, 1, 0, 1, 0, 8, \ldots\}$

($a'_j = 0$ when j is even). Thus $S_m = O(m^2)$.

Things go differently for $e/2$:

$$e/2 = [1; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, \ldots]$$

Why $S_N(e)$ behaves differently from $S_N(2e)$?

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Asymptotic upper bounds for $D_N(\alpha, 1/2)$ depends on $S_m = \sum_{j=1}^{m} a'_j$.

For $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ we have $\{a'_j\}_{j\geq 1} = \{1, 0, 1, 0, 4, 0, 1, 0, 1, 0, 8, \ldots\}$

($a'_j = 0$ when j is even). Thus $S_m = O(m^2)$.

Things go differently for $e/2$:

$e/2 = [1; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, \ldots]$
$\{a'_j\}_{j\geq 1} = \{2, 0, 3, 0, 0, 0, 3, 0, 3, 0, 0, 0, 3, 0, 3, 0, 0, \ldots\}$.

Why $S_N(e)$ behaves differently from $S_N(2e)$?

More precisely, let p_n/q_n be the convergents of α and put $a'_{j+1} = a_{j+1}$ if $q_j \equiv j \equiv 0 \mod 2$, and $a'_{j+1} = 0$ otherwise.

Asymptotic upper bounds for $D_N(\alpha, 1/2)$ depends on $S_m = \sum_{j=1}^{m} a'_j$.

For $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ we have $\{a'_j\}_{j\geq 1} = \{1, 0, 1, 0, 4, 0, 1, 0, 1, 0, 8, \ldots\}$

($a'_j = 0$ when j is even). Thus $S_m = O(m^2)$.

Things go differently for $e/2$:

$$e/2 = [1; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, \ldots]$$
$$\{a'_j\}_{j\geq 1} = \{2, 0, 3, 0, 0, 0, 3, 0, 3, 0, 0, 0, 3, 0, 3, 0, 0, \ldots\}.$$

($a'_j = 0$ when $j \equiv 0, 2, 4, 5 \mod 6$). Thus now $S_m = O(m)$.

Concluding remarks

We can go further (with "non classical" numbers):

Concluding remarks

We can go further (with “non classical” numbers):

Given $\delta \colon \mathbb{N} \to \mathbb{R}^+$ which tends to infinity, there exists $\alpha \notin \mathbb{Q}$ such that $D_N(\alpha, 1/2) = O(\delta(N))$.

Concluding remarks

We can go further (with "non classical" numbers):

Given $\delta\colon \mathbb{N} \to \mathbb{R}^+$ which tends to infinity, there exists $\alpha \notin \mathbb{Q}$ such that $D_N(\alpha, 1/2) = O(\delta(N))$.

Question (Waldschmidt, tgv Paris-Turin)

How are these kinds of phenomena "rare"?