The Tiling Problem using simple and multidimensional continued fractions

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Outline

Continued Fractions

Continued Fraction

Definition

A continued fraction *is an expression of the form*

$$[a_0, (b_1, a_1), (b_2, a_2), (b_3, a_3), \dots] := a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}}$$

The coefficients a_i are called partial denominators*, while the values b_i are called* partial numerators*. If $b_i = 1$ for all $i \geq 1$, the continued fraction is called* simple*.*

Convergents

Definition

If we consider the finite expansion

$$[a_0, (b_1, a_1), (b_2, a_2), \dots, (b_n, a_n)] = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{\ddots + \cfrac{b_n}{a_n}}}$$

this is equal to a value $\frac{A_n}{B_n}$*, called* $n-$**th convergent***.*

Convergents

Example

Consider the continued fraction

$$\alpha = [2, (1, 1), (1, 2), (2, 3), (3, 4), \ldots]$$

The first convergents of α are

$$\alpha_0 = 2; \qquad \alpha_1 = 2 + \frac{1}{1} = 3;$$

$$\alpha_2 = 2 + \cfrac{1}{1 + \cfrac{1}{2}} = \frac{8}{3} = 2,\overline{6};$$

$$\alpha_3 = \frac{30}{11} = 2,\overline{72};$$

$$\alpha_4 = \frac{144}{53} = 2,71698; \qquad \alpha_5 = \frac{280}{103} = 2,71844.$$

Convergents

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Consider the continued fraction

$$\alpha = [2, (1,1), (1,2), (2,3), (3,4), \ldots] = e = 2,71828.$$

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Fundamental recurrence formulas

The numerator A_n and the denominator B_n of the n-th convergent can be recovered using the following recurrences:

$$\begin{cases} A_{-1} &= 1, \\ A_0 &= a_0, \\ A_n &= a_n A_{n-1} + b_n A_{n-2}. \end{cases} \qquad \begin{cases} B_{-1} &= 0, \\ B_0 &= 1, \\ B_n &= a_n B_{n-1} + b_n B_{n-2}. \end{cases}$$

Theorem (Determinant formula)

The numerators and denominators of the convergents satisfy

$$A_{n-1}B_n - A_n B_{n-1} = \prod_{i=1}^{n}(-b_i) = (-1)^n \prod_{i=1}^{n} b_i,$$

for any $n \geq 1$.

Properties of simple continued fractions

Given a number $\alpha \in \mathbb{R}^+$, the simple continued fraction expansion of α is given by

$$\begin{cases} \alpha_0 & = \alpha; \\ a_n & = \lfloor \alpha_n \rfloor; \\ \alpha_{n+1} & = \frac{1}{\alpha_n - a_n}. \end{cases}$$

The resulting continued fraction is

$$\alpha = [a_0, (1, a_1), (1, a_2), \ldots] = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

Properties of simple continued fractions

Example

Consider $\Phi = \frac{1+\sqrt{5}}{2}$. Using the previous algorithm we may compute the partial quotients:

$$\begin{cases} \alpha_0 = \Phi; \\ a_0 = \lfloor \Phi \rfloor = 1; \\ \alpha_1 = \left(\frac{1+\sqrt{5}}{2} - 1\right)^{-1} = \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{5-1} = \Phi, \end{cases}$$

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}$$

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The sequence of the numerators of the convergents can be recovered by the recursion:

$$\begin{cases} A_{-1} = 1, \\ A_0 = 1, \\ A_n = A_{n-1} + A_{n-2}, \text{ for any } n \geq 1, \end{cases}$$

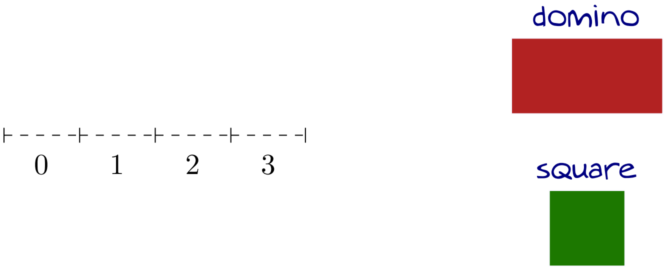
that is the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$.

The Tiling Problem

A "simple" version

A first problem

Suppose there is a table of dimension $1 \times (n+1)$ and imagine to have only squares and dominoes of length respectively 1×1 and 1×2 to fully cover the table.



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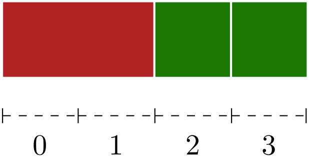
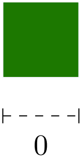
How many tilings are there?



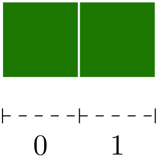
Counting some tilings

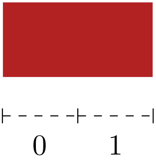
- $n = 0$



Counting some tilings

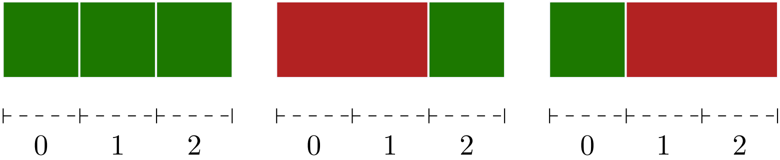
- $n = 1$





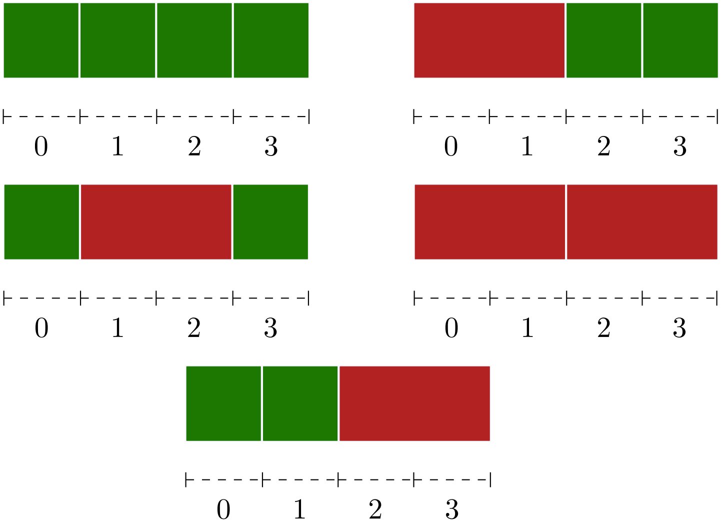
Counting some tilings

- $n = 2$



Counting some tilings

- $n = 3$



Counting some tilings

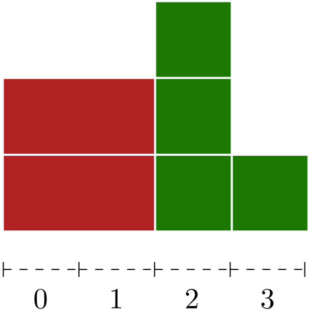
Dimension	Number of tilings
$n = 0$	1
$n = 1$	2
$n = 2$	3
$n = 3$	5
$n = 4$	8
$n = 5$	13

The stackable Tiling Problem

The height conditions

Suppose that tiles of the same dimension can be stacked such that

- at most a_i squares can be stacked in cell i for any $0 \leq i \leq n$;
- at most b_i dominoes can be stacked in cells $i-1, i$ for any $1 \leq i \leq n$.



The picture on the left satisfies the following height condition:

$$[a_0, (b_1, a_1), (b_2, a_2), (b_3, a_3)] =$$
$$= [1, (2, 1), (1, 5), (3, 3)].$$

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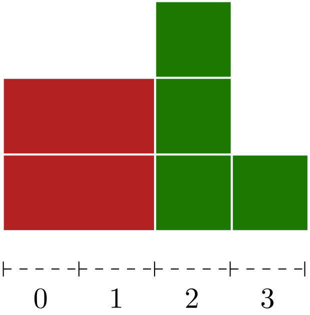

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Continued fractions and the Tiling Problem

Theorem (Benjamin and Quinn, 2003)

Let $n \in \mathbb{N}^+$, *then the number of stackable tilings of a table of length* $(n+1)$ *with height conditions* $[a_0, (b_1, a_1), \dots, (b_n, a_n)]$ *is equal to* A_n, *the numerator of the* n*-th covergent, while the denominator* B_n *counts the number of stackable tilings of a table of length* n *with height conditions* $[a_1, (b_2, a_2), \dots, (b_n, a_n)]$.

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Example

If we may use at most just one square or one domino, we have the height conditions $[1, (1, 1), \ldots (1, 1)]$, which correspond to the convergents of the continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{\ddots}} = \frac{1 + \sqrt{5}}{2}.$$

Tilings and permutations

As we already noticed,

$$e = [2, (1,1), (1,2), (2,3), \ldots].$$

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- there exists a bijection between stackable tilings of length $(n+1)$ with $[2, (1, 1), (1, 2), \dots, (n-1, n)]$ as height conditions and permutations in S_n and S_{n+1};

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- there exists a bijection between stackable tilings of length n with $[1, (1,2), \dots, (n, n-1)]$ as height conditions and scramblings in S_{n+1} (scaramblings are permutations having no fixed adjacencies).

Mixed Tilings

Negative dominoes

Mixed tilings [Eustis, 2006]

Suppose that $a_i > 0$ for any $0 \leq i \leq n$ and let $b_j \in \mathbb{Z}^*$ for any $1 \leq j \leq n$ such that $a_k > -b_k$ for any $1 \leq k \leq n$. Then, for any $1 \leq j \leq n$,

- if b_j is positive, then the height conditions are the same as before;
- if b_j is negative, then we cannot use dominoes to cover cells $j-1$ and j and we have to discard as invalid all the tilings in which in cell $j-1$ there is a stack of full a_{j-1} squares and that have $|b_j|$ squares or less in cell j.

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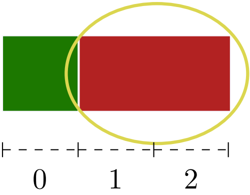
How many mixed tilings are there?

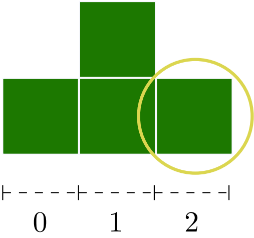
Negative dominoes

Example

Consider the following height conditions:

$$[a_0, (a_1, b_1), (a_2, b_2)] = [1, (1, 2), (-1, 3)].$$





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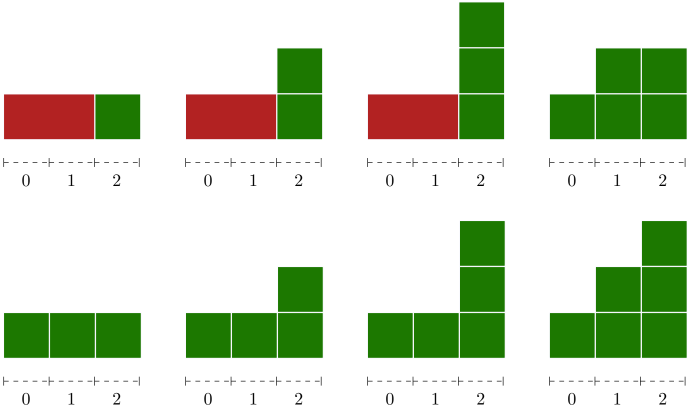

Number of mixed tilings

Theorem (Eustis, 2006)

Let $n \in \mathbb{N}^+$, then the number of mixed tilings of a table of length $(n+1)$ with height conditions $[a_0, (b_1, a_1), \dots, (b_n, a_n)]$ is equal to A_n, the numerator of the n-th covergent, while the denominator B_n counts the number of mixed tilings of a table of length n with height conditions $[a_1, (b_2, a_2), \dots, (b_n, a_n)]$.

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Example

In the previous example,

$$[1, (1,2), (-1,3)] = 1 + \cfrac{1}{2 - \cfrac{1}{3}} = \frac{8}{5}.$$

Multidimensional Continued Fractions

Muldidimensional Continued Fraction

Definition

A multidimensional continued fraction *is an expression of the form*

$$\alpha_0 = a_0 + \cfrac{b_1 + \cfrac{c_2}{a_2 + \cfrac{}{\ddots}}}{a_1 + \cfrac{b_2 + \cfrac{c_3}{\ddots}}{a_2 + \cfrac{}{\ddots}}}, \qquad \beta_0 = b_0 + \cfrac{c_1}{a_1 + \cfrac{b_2 + \cfrac{c_3}{\ddots}}{a_2 + \cfrac{}{\ddots}}}$$

Muldidimensional Continued Fraction

Definition

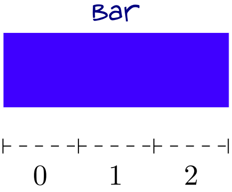
- *If* $c_i = 1$ *for any* $i \geq 1$*, the multidimensional continued fraction is called* **simple**.
- *A multidimensional continued fraction is denoted as* $[(a_0, \ldots), (b_0, \ldots), (c_1, \ldots)]$.
- *The finite expansion* $[(a_0, \ldots, a_n), (b_0, \ldots, b_n), (c_1, \ldots, c_n)]$ *is equal to the couple* $\left(\frac{A_n}{B_n}, \frac{C_n}{B_n}\right)$*, called* **$n$−th convergent**.

The Multidimensional Tiling Problem

joint work with M. Battagliola and N. Murru

A new tile

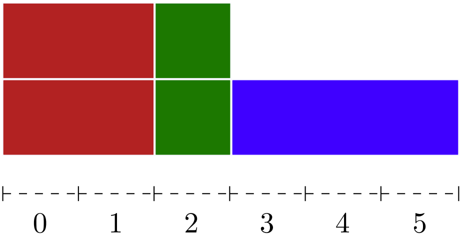
Suppose that we add a new shape in the Tiling Problem of dimension 1×3, called a Bar.



Stackable Multidimensional Tiling Problem

Suppose that tiles of the same dimension can be stacked such that

- at most a_i squares can be stacked in cell i for any $0 \leq i \leq n$;
- at most b_i dominoes can be stacked in cells $i-1, i$ for any $1 \leq i \leq n$.
- at most c_i bars can be stacked in cells $i-2, i-1, i$ for any $2 \leq i \leq n$.



The picture on the left satisfies the following height condition:

$$[(1, 2, 2, 3, 1, 1), (2, 1, 3, 1, 2), (2, 1, 2, 1)].$$

A more complex Tiling Problem

Stackable Multidimensional Tiling Problem

Suppose that tiles of the same dimension can be stacked such that

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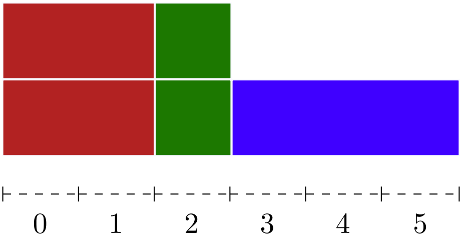
How many tilings are there?



The picture on the left satisfies the following height condition:

$$[(1,2,2,3,1,1),(2,1,3,1,2),(2,1,2,1)].$$

Continued fractions and the Tiling Problem

Theorem

Let $n \in \mathbb{N}^+$, then the number of stackable tilings of a table of length $(n+1)$ with height conditions

$$[(a_0, a_1, \ldots, a_n), (b_0, b_1, \ldots, b_n), (1, c_1, \ldots, c_n)]$$

is equal to A_n, the first numerator of the n-th covergent, while the denominator B_n counts the number of stackable tilings of a table of length n with height conditions $[(a_1, \ldots, a_n), (b_2, \ldots, b_n), (c_3, \ldots, b_n)]$.

Remark

Note that b_0 and c_1 do not appear in the first convergent.

The second numerator

Theorem

Let $n \in \mathbb{N}^+$, then the number of stackable tilings of a table of length $(n+2)$[a] *with height conditions $[(a_0, a_1, \dots, a_n), (b_0, b_1, \dots, b_n), (1, c_1, \dots, c_n)]$, such that the first tile of the tiling is a stack of dominoes or bars is equal to C_n, the second numerator of the n-th convergent.*

[a] to be consistent with the notation, in this case the first cell is labelled with -1 (i.e., we add a cell on the left to a $(n+1)$–board).

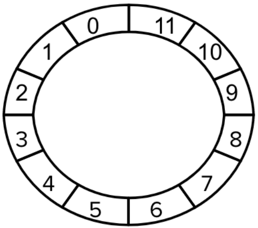
Remark

Note that a_0 does not appear in the second convergent.

The Circular Tiling Problem

Circular Tiling

A circular table of dimension $1 \times (n+1)$, is a $(n+1)$–table where the first and last tile are bordering.
Compared to the previous problem there are additional tilings, in particular there are also tiling with dominoes or bars covering the first and the last cell.



Example of a circular board with $n = 11$.

Number of circular tilings

Theorem

The number of tilings of a $(n+1)$*–circular table with height condition* $[(a_0, a_1, \ldots, a_n), (b_0, b_1, \ldots, b_n), (1, c_1, \ldots, c_n)]$ *with* $c_0 = 0$ *(i.e. we forbid bars covering the cells* $0, n, n-1$*) is* $A_n + C_{n-1}$*.*

Theorem

If we consider the following height conditions

$$[(4, 2, 3, 4, ...), (1, 1, 2, 3, ...), (1, 1, 1, 2, ...)],$$

then

$$A_n = (n + 2)! + (n + 1)! + n!$$

Multidimensional Tilings and permutations

Theorem

If we consider the following height conditions

$$[(4, 2, 3, 4, ...), (1, 1, 2, 3, ...), (1, 1, 1, 2, ...)],$$

then

$$A_n = (n+2)! + (n+1)! + n!$$

Remark

The MCF of the previous proposition is

$$[(4, 2, 3, 4, 5, 6, \dots), (b_0, 1, 1, 2, 3, 4, \dots), (1, c_1, 1, 1, 2, 3, \dots)]$$

and the first sequence of convergents $\left(\frac{A_n}{B_n}\right)_{n\geq 0}$ appears to be convergent to the real number $4.54752\dots$, but we were not able to explicitly determine this real number.

Negative height conditions

Mixed Multidimensional Tiling Problem

Let $(a_i)_{i \geq 0}$ be a sequence of positive integers and $(b_i)_{i \geq 0}$, $(c_i)_{i \geq 0}$ be sequences of integers such that

- if $b_i < 0$ and $c_i > 0$, then $a_i > |b_i|$;
- if $b_i > 0$ and $c_i < 0$, then either $a_i > |c_i|$ or $b_i > |c_i|$;
- if $b_i < 0$ and $c_i < 0$, then $a_i > |b_i| + |c_i|$.

Negative dominoes

Mixed Multidimensional Tiling Problem

We define a **mixed tiling** of an $(n+1)$–table with height condition respectively given by $[(a_0, a_1, \ldots, a_n), (b_0, b_1, \ldots, b_n), (1, c_1, \ldots, c_n)]$ as follows:

for any $k \in \mathbb{N}$,

1. if $b_k \geq 0$ and $c_k \geq 0$, we fall back in the same case defined before;
2. if $b_k < 0$ and $c_k > 0$, when there is a stack of a_{k-1} squares in the cell $k-1$, we discard the tilings having up to $|b_k|$ squares in the cell k;

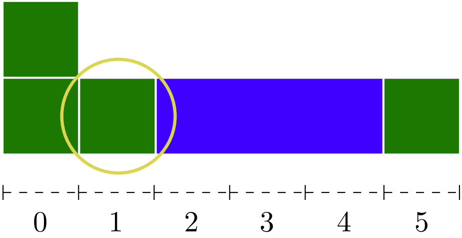
Negative dominoes

Example

Consider the height conditions given by the following continued fraction:

$$(\boldsymbol{a} = [2, 3, 1, 2, 2, 3], \boldsymbol{b} = [-, -1, 3, 3, 2, -1], \boldsymbol{c} = [-, -, -2, 2, 1, -1]).$$

2. Since $b_1 = -1$, then when we have $a_0 = 2$ squares in position 0, we need to exclude the case having 1 square in the tile in position 1.



Negative Bars

Mixed Multidimensional Tiling Problem

3. if $c_k < 0$ and $b_k > 0$, we have two exclusive cases:
   a. if $a_k > |c_k|$, when there is a stack stack of a_{k-2} squares in the cell $k-2$ and a stack of a_{k-1} squares in the cell $k-1$, then we consider as invalid all the tilings having up to $|c_k|$ squares in the cell k;
   b. otherwise, necessarily $b_k > |c_k|$. In this case when there is a stack of a_{k-2} squares in the cell $k-2$, the invalid tilings are those with up to $|c_k|$ dominoes covering the cells $k-1, k$;

Negative Bars

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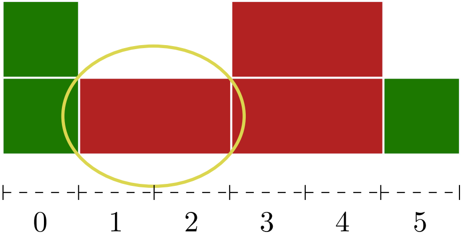
Consider the height conditions given by the following continued fraction:

$$(\boldsymbol{a} = [2, 3, 1, 2, 2, 3], \boldsymbol{b} = [-, -1, 3, 3, 2, -1], \boldsymbol{c} = [-, -, -2, 2, 1, -1]).$$

3b Since $c_2 = -2$ and $a_2 = 1 < |c_2|$ we need to exclude the tilings having 2 squares in position 0 and 1 or 2 dominoes in the positions 1 and 2.



Negative dominoes and negative bars

Mixed Multidimensional Tiling Problem

4. if $c_k < 0$ and $b_k < 0$, we have two non-exclusive cases:
   a. when at the same time there is a stack of a_{k-2} squares in the cell $k-2$ and a stack of a_{k-1} squares in the cell $k-1$, we discard all the tilings having up to $|c_k| + |b_k|$ squares in the cell k;
   b. when there is a stack of a_{k-1} squares in the cells $k-1$, the invalid tilings have up to $|b_k|$ squares in the cell k.

Remark

Note that the case (4b) applies independently on the number of squares in cell $k-2$.

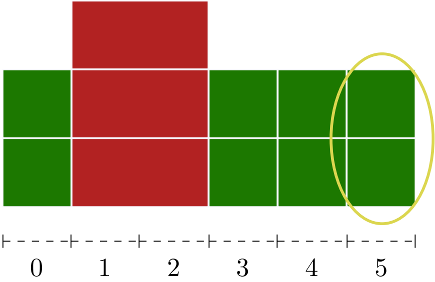
Negative dominoes and negative bars

Example

Consider the height conditions given by the following continued fraction:

$$(\boldsymbol{a} = [2, 3, 1, 2, 2, 3], \boldsymbol{b} = [-, -1, 3, 3, 2, -1], \boldsymbol{c} = [-, -, -2, 2, 1, -1]).$$

(4a) Since $b_5 = c_5 = -1$, the inadmissible tilings are those having $a_3 = 2$ squares in position 3, $a_4 = 2$ squares in position 4 and 1 or 2 squares in position 5.



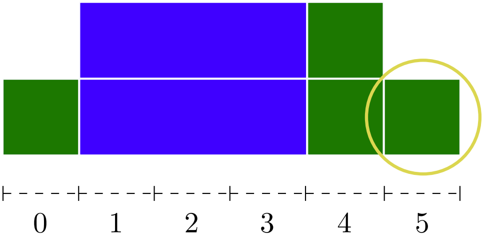
Negative dominoes and negative bars

Example

Consider the height conditions given by the following continued fraction:

$$(\boldsymbol{a} = [2, 3, 1, 2, 2, 3], \boldsymbol{b} = [-, -1, 3, 3, 2, -1], \boldsymbol{c} = [-, -, -2, 2, 1, -1]).$$

(4b) Moreover we also need to discard the tilings having $a_4 = 2$ squares in position 4 and $|b_5| = 1$ square in position 5.



An answer to the Multidimensional Tiling Problem

Theorem

Let $n \in \mathbb{N}^+$, then the number of mixed multidimensional tilings of a table of length $(n+1)$ with height conditions $[(a_0, a_1, \ldots, a_n), (b_0, b_1, \ldots, b_n), (c_0, c_1, \ldots, c_n)]$ is equal to A_n, the numerator of the first n-th covergent.

THANK YOU FOR THE ATTENTION!

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