Explicit bounds for a generating set of the class group (under GRH)

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Let $\mathbb{K}/\mathbb{Q}$ be a number field, with $n = [\mathbb{K} : \mathbb{Q}]$ the degree and r_1, r_2 the real (respectively pair of) complex embedding of $\mathbb{K}$ (so that $n = r_1 + 2r_2$). Let Δ be the absolute value of the absolute discriminant of $\mathbb{K}$. Let $\mathcal{Cl}_{\mathbb{K}}$ be the class group for $\mathbb{K}$, i.e. the quotient of the group of fractional ideals in $\mathbb{K}$ by the subgroup of principal fractional ideals.

It is known that $\mathcal{Cl}_{\mathbb{K}}$ is a finite and abelian group. Buchmann's algorithm is an efficient method to compute $\mathcal{Cl}_{\mathbb{K}}$, but it needs as basic ingredient a list of generators. Let $T_{\mathbb{K}}$ be the minimum of integers T such that $\{[\mathfrak{p}] : \mathfrak{p} \text{ prime}, \mathrm{N}\mathfrak{p} \leq T\}$ is a generating set for $\mathcal{Cl}_{\mathbb{K}}$. Main problem: how to estimate $T_{\mathbb{K}}$?

Minkowski (1910?): $T_{\mathbb{K}} \leq \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n}\sqrt{\Delta} \lesssim (7.38^{r_1/2} \cdot 5.8^{r_2})^{-1}\sqrt{\Delta}$ *as* $n \to \infty$.

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Zimmert ('81): $T_{\mathbb{K}} \lesssim (50.7^{r_1/2} \cdot 19.9^{r_2})^{-1}\sqrt{\Delta}$ *as* $n \to \infty$.

de la Meza (2002): *Better constants for a lot of signatures for* $n \leq 10$, *but still* $\sqrt{\Delta}$.

Problem: Δ can be extremely large also for not so 'strange' fields, the bound becomes too large.

Class group: preliminary bounds for generators

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Conditional explicit bounds

Theorem (Bach '90): *(GRH)* $T_{\mathbb{K}} \leq \begin{cases} 12\log^2\Delta \\ (4+o(1))\log^2\Delta & \textit{as } \Delta\to\infty \end{cases}$

($o(1)$ term not explicit but the method cannot prove that it is lower than $(\log\Delta)^{-2/3}$).

Reworking the proof of this result, Belabas, Diaz y Diaz and Friedman (2008) defined a (GRH) algorithm which in all test has been able to produce an even shorter generating set, but that actually estimates $T_{\mathbb{K}}$ with T_{BD^2F} where

$$\big(\log\Delta\log\log\Delta\big)^2 \ll T_{BD^2F} \leq \Big(\big(\frac{1}{4}+o(1)\big)\log\Delta\log\log\Delta\Big)^2.$$

Hence the very good performance of the algorithm is due to the small value for the constant in front and the very slow increasing behaviour of log log: however, in the (very long) run (essentially $\Delta \geq \exp(10^6)$) it is worse than Bach's bound.

Reworking one more time the approach:

Theorem (Grenié-M. 2018) *(GRH)* $T_{\mathbb{K}} \leq \begin{cases} 4.01\log^2\Delta \\ 4(\log\Delta+\log\log\Delta+1-2n)^2 \end{cases}$

(the second with the exceptions $\mathbb{K}=\mathbb{Q}[\sqrt{D}]$ for $D\in\{-15,-11,-8,-7,8,12\}$).

Also this approach can be modified to produce a (GRH) algorithm: in all tests the new algorithm produces a generating set supporting the conjecture that

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Forward

$$\delta_{\chi_0}(\phi(1)+\phi(0)) - \sum_{\rho\in L_\chi}\phi(\rho) = \sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})}{\sqrt{\mathrm{N}\mathfrak{a}}}(\chi(\mathfrak{a})+\overline{\chi(\mathfrak{a})})F(\log \mathrm{N}\mathfrak{a})$$

$$+ F(0)(n(\gamma+\log 2\pi)-\log\Delta) + r_1\int_0^{+\infty}\frac{F(x)}{2\,\mathrm{Ch}(x/2)}\,\mathrm{d}x + n\int_0^{+\infty}\frac{F(x)-F(0)}{2\,\mathrm{Sh}(x/2)}\,\mathrm{d}x.$$

Let $T < T_{\mathbb{K}}$. Then the prime ideals with norm $\le T$ generate a proper subgroup of $\mathcal{C}\ell_{\mathbb{K}}$. Hence there exists $\chi \neq \chi_0$ which is trivial in the subgroup. Suppose that F is supported in $[-L, L]$ with $L := \log T$. Subtract the formulas for $L(s,\chi_0) = \zeta_{\mathbb{K}}$ and $L(s,\chi)$:

$$\phi(1)+\phi(0) - \sum_{\rho\in\zeta_{\mathbb{K}}}\phi(\rho) + \sum_{\rho\in L_\chi}\phi(\rho) = 0.$$

Suppose that $\phi(1/2+it) \ge 0$ for every $t\in\mathbb{R}$ and GRH: then $\sum_{\rho\in L_\chi}\phi(\rho)\ge 0$ and the formula becomes

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$$\delta_{\chi_0}(\phi(1)+\phi(0)) - \sum_{\rho\in L_\chi}\phi(\rho) = \sum_{\mathfrak{a}}\frac{\Lambda(\mathfrak{a})}{\sqrt{\mathrm{N}\mathfrak{a}}}(\chi(\mathfrak{a})+\overline{\chi(\mathfrak{a})})F(\log\mathrm{N}\mathfrak{a})$$
$$+F(0)(n(\gamma+\log 2\pi)-\log\Delta) + r_1\int_0^{+\infty}\frac{F(x)}{2\,\mathrm{Ch}(x/2)}\,\mathrm{d}x + n\int_0^{+\infty}\frac{F(x)-F(0)}{2\,\mathrm{Sh}(x/2)}\,\mathrm{d}x.$$

Let $T < T_{\mathbb{K}}$. Then the prime ideals with norm $\leq T$ generate a proper subgroup of $\mathcal{C}\ell_{\mathbb{K}}$. Hence there exists $\chi \neq \chi_0$ which is trivial in the subgroup. Suppose that F is supported in $[-L, L]$ with $L := \log T$. Subtract the formulas for $L(s,\chi_0) = \zeta_{\mathbb{K}}$ and $L(s,\chi)$:

$$\phi(1)+\phi(0) - \sum_{\rho\in\zeta_{\mathbb{K}}}\phi(\rho) + \sum_{\rho\in L_\chi}\phi(\rho) = 0.$$

Suppose that $\phi(1/2+it) \geq 0$ for every $t\in\mathbb{R}$ and GRH: then $\sum_{\rho\in L_\chi}\phi(\rho)\geq 0$ and the formula becomes

$$\phi(1)+\phi(0) \leq \sum_{\rho\in\zeta_{\mathbb{K}}}\phi(\rho).$$

Thus, in case

$$4\int_0^{+\infty}F(x)\,\mathrm{Ch}(x/2)\,\mathrm{d}x = \phi(1)+\phi(0) > \sum_{\rho\in\zeta_{\mathbb{K}}}\phi(\rho) \qquad \text{then} \qquad T_{\mathbb{K}} \leq T.$$

Since $\phi(1/2+it) = \int_{\mathbb{R}} F(x)e^{itx}\,\mathrm{d}x = \hat{F}(t)$, we can meet the condition $\phi(1/2+it) \geq 0$ setting $F = \psi * \psi$ where ψ is real, even, stepwise C^1 and supported in $[-L/2, L/2]$. In terms of ψ the criterion becomes:

$$\begin{cases} \text{if } \exists\psi \text{ as above and with} \\ 8\Big[\int_{\mathbb{R}} \psi(x)\,\mathrm{Ch}(x/2)\Big]^2 > \sum_{\gamma\in\zeta_{\mathbb{K}}} |\hat{\psi}(\gamma)|^2 \end{cases} \implies T_{\mathbb{K}} \leq T.$$

We further specialize the function setting $\psi = \psi^+(L/2+x) + \psi^+(L/2-x)$, where ψ^+ is positive and supported in $[0, L]$. Then

$$2\int_{\mathbb{R}} \psi(x)\,\mathrm{Ch}(x/2)\,\mathrm{d}x = T^{-\frac{1}{4}}\int_0^L \psi^+(x)e^{\frac{x}{2}}\,\mathrm{d}x + T^{\frac{1}{4}}\int_0^L \psi^+(x)e^{-\frac{x}{2}}\,\mathrm{d}x.$$

Moreover, $\hat{\psi}(t) = 2\,\mathrm{Re}[e^{-iLt/2}\widehat{\psi^+}(t)]$ so that $|\hat{\psi}(t)|^2 \leq 4|\widehat{\psi^+}(t)|^2 = 4\widehat{\psi^+ * \psi^-}(t)$ where $\psi^-(x) := \psi^+(-x)$, and the criterion becomes:

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The key point here is that the series on zeros involves values of a Fourier trasform, so that we can apply once more time the explicit formula for $\zeta_{\mathbb{K}}$ with $F = 4\psi^+ * \psi^-$ to write this sum as a sum on ideals ◂ Weil-Poitou , getting that:

Suitable choice for F

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$$-8\sum_{\mathfrak{a}}\frac{\Lambda(\mathfrak{a})}{\sqrt{\mathrm{N}\mathfrak{a}}}(\psi^+ * \psi^-)(\log \mathrm{N}\mathfrak{a}) + 4(\psi^+ * \psi^-)(0)(\log\Delta - (\gamma + \log 2\pi)n)$$

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with $I(g) := \int_0^{+\infty}\frac{g(x)-g(0)}{\mathrm{Sh}(x/2)}\,\mathrm{d}x$. After some book-keeping we get

$$\sqrt{T} > 2T\frac{\int_0^L(\psi^+)^2(x)\,\mathrm{d}x}{\big(\int_0^L\psi^+(x)e^{\frac{x}{2}}\,\mathrm{d}x\big)^2}(\log\Delta - (\gamma + \log 8\pi)n)$$

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