Maxima of polynomials and small regulators

Francesco Battistoni
(joint work with Giuseppe Molteni)

Università degli Studi di Milano

6th Number Theory Meeting, Torino
22/09/2022

Classification of number fields with small regulators

Astudillo, Diaz y Diaz, Friedman (2016): a procedure which classifies number fields with fixed signature (r_1, r_2) (number of real and complex places) and small regulator starting from lists of fields with small discriminant.

* Fields of degree ≤ 6.
* Fields of degree 7 excluding signature $(5, 1)$.
* Totally real fields of degree 8 and 9.
* Totally complex fields of degree 8.

Classification of number fields with small regulators

Astudillo, Diaz y Diaz, Friedman (2016): a procedure which classifies number fields with fixed signature (r_1, r_2) (number of real and complex places) and small regulator starting from lists of fields with small discriminant.

* Fields of degree ≤ 6.
* Fields of degree 7 excluding signature $(5, 1)$.
* Totally real fields of degree 8 and 9.
* Totally complex fields of degree 8.

Friedman, Ramirez-Raposo (2018): classification in signature $(5, 1)$ with an ad hoc improvement.
Next case of interest is degree 8, signature $(6, 1)$.
Problem: the procedure fails (just like it failed for $(5, 1)$) because the upper bound of a key inequality is too big.

The key estimate

The procedure needs an estimate for

$$P_n(\varepsilon_1, \dots, \varepsilon_n) := \prod_{1 \leq i < j \leq n} \left| 1 - \frac{\varepsilon_i}{\varepsilon_j} \right|$$

where the ε_i are complex numbers with the assumption

$$0 < |\varepsilon_1| \leq |\varepsilon_2| \leq \cdots |\varepsilon_n|.$$

The key estimate

The procedure needs an estimate for

$$P_n(\varepsilon_1, \ldots, \varepsilon_n) := \prod_{1 \le i < j \le n} \left| 1 - \frac{\varepsilon_i}{\varepsilon_j} \right|$$

where the ε_i are complex numbers with the assumption

$$0 < |\varepsilon_1| \le |\varepsilon_2| \le \cdots |\varepsilon_n|.$$

Remak (1952): $P_n \le n^{n/2}$.
Pohst (1977): $P_n \le 2^{\lfloor n/2 \rfloor}$ if ε_i all real and $n \le 11$.
Idea: If signature is (r_1, r_2), we should estimate P_n with r_2 couples of conjugated numbers among the ε_i (and the remaining are all real).

The key estimate

The procedure needs an estimate for

$$P_n(\varepsilon_1, \dots, \varepsilon_n) := \prod_{1 \le i < j \le n} \left| 1 - \frac{\varepsilon_i}{\varepsilon_j} \right|$$

where the ε_i are complex numbers with the assumption

$$0 < |\varepsilon_1| \le |\varepsilon_2| \le \cdots |\varepsilon_n|.$$

Remak (1952): $P_n \le n^{n/2}$.
Pohst (1977): $P_n \le 2^{\lfloor n/2 \rfloor}$ if ε_i all real and $n \le 11$.
Idea: If signature is (r_1, r_2), we should estimate P_n with r_2 couples of conjugated numbers among the ε_i (and the remaining are all real).

Theorem (B.-Molteni (2021-22))

* $P_n \le 2^{\lfloor n/2 \rfloor}$ for $(n, 0)$ and every $n \in \mathbb{N}$. (This bound is achieved).
* $P_n \le 16.6965\dots$ for $(3, 1)$. (This bound is achieved).
* $P_n \le 34.89$ for $(4, 1)$, $P_n \le 65.81$ for $(5, 1)$, $P_n \le 83.9$ for $(6, 1)$ and $P_n \le 233.1$ for $(7, 1)$.

Totally real case

We take P_{n+1} and set $x_i := \varepsilon_i/\varepsilon_{i+1}$.

$$Q_n(x_1, \dots, x_n) := \prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \prod_{k=i}^{j} x_k\right), \quad x_i \in [-1, 1], \quad M_n := \max_{[-1,1]^n} Q_n.$$

It is easy to prove that $M_1 = M_2 = 2$ by standard optimization. But as n grows, this local process becomes unfeasible.

Totally real case

We take P_{n+1} and set $x_i := \varepsilon_i/\varepsilon_{i+1}$.

$$Q_n(x_1, \dots, x_n) := \prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \prod_{k=i}^{j} x_k\right), \quad x_i \in [-1, 1], \quad M_n := \max_{[-1,1]^n} Q_n.$$

It is easy to prove that $M_1 = M_2 = 2$ by standard optimization. But as n grows, this local process becomes unfeasible.

Idea: given a vector of signs $\boldsymbol{v} := (v_1, \dots, v_n)$, we consider

$$Q_{n,\boldsymbol{v}}(x_1, \dots, x_n) := \prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - \prod_{k=i}^{j} v_k \prod_{k=i}^{j} x_k\right), \qquad x_i \in [0, 1]$$

which we call a **configuration of** Q_n.

Totally real case

We take P_{n+1} and set $x_i := \varepsilon_i/\varepsilon_{i+1}$.

$$Q_n(x_1,\dots,x_n) := \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\prod_{k=i}^{j}x_k\right), \quad x_i\in[-1,1], \quad M_n := \max_{[-1,1]^n} Q_n.$$

It is easy to prove that $M_1 = M_2 = 2$ by standard optimization. But as n grows, this local process becomes unfeasible.

Idea: given a vector of signs $\boldsymbol{v} := (v_1,\dots,v_n)$, we consider

$$Q_{n,\boldsymbol{v}}(x_1,\dots,x_n) := \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\prod_{k=i}^{j}v_k\prod_{k=i}^{j}x_k\right), \qquad x_i\in[0,1]$$

which we call a **configuration of** Q_n.

If we can prove $Q_{n,\boldsymbol{v}} \le 2^{\lfloor \frac{n+1}{2}\rfloor}$ for the 2^n choices of $\boldsymbol{v}$, we obtain the desired value for M_n.

$$\begin{aligned} Q_{3,(+,-,-)}(x_1,x_2,x_3) = \quad &(1-x_1) \quad (1+x_1x_2) \quad (1-x_1x_2x_3)\\ &\qquad\qquad\; (1+x_2) \qquad\; (1-x_2x_3)\\ &\qquad\qquad\qquad\qquad\qquad\;\; (1+x_3)\end{aligned}$$

Graphical schemes

We can represent a configuration $Q_{n,\boldsymbol{v}}$ with a triangular array formed by signs $+$ and $-$, each sign at (i,j) being equal to $\prod_{k=i}^{j} v_k$.

$$Q_{3,(+,-,-)} = \begin{array}{ccc} (1-x_1) & (1+x_1x_2) & (1-x_1x_2x_3) \\ & (1+x_2) & (1-x_2x_3) \\ & & (1+x_3) \end{array} = \begin{array}{|c|c|c|} \hline + & - & + \\ \hline \multicolumn{1}{c|}{} & - & + \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & - \\ \cline{3-3} \end{array}$$

Graphical schemes

We can represent a configuration $Q_{n,\boldsymbol{v}}$ with a triangular array formed by signs $+$ and $-$, each sign at (i,j) being equal to $\prod_{k=i}^{j} v_k$.

$$Q_{3,(+,-,-)} = \begin{array}{ccc} (1-x_1) & (1+x_1x_2) & (1-x_1x_2x_3) \\ & (1+x_2) & (1-x_2x_3) \\ & & (1+x_3) \end{array} = \begin{array}{|c|c|c|} \hline + & - & + \\ \hline & - & + \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & - \\ \cline{3-3} \end{array}$$

Every $n \times n$ triangular array C formed by $+$ and $-$ (we call it **graphical scheme of dimension** n) corresponds to a function $F_C : [0,1]^n \to \mathbb{R}$ defined as

$$F_C(x_1, \dots, x_n) = \prod_{i=1}^{n} \prod_{j=i}^{n} \left(1 - C_{i,j} \prod_{k=i}^{j} x_k \right).$$

Instead of studying configurations analytically, we look at their associated graphical schemes.

Estimates and patterns of graphical schemes

Pohst's original idea: in a graphical scheme C we can recognize patterns, corresponding to bounded factors of F_C.

$i \overset{j}{\boxed{+}} \leq 1$ since it corresponds to $(1-u) \leq 1$ for $u \in [0,1]$.

Estimates and patterns of graphical schemes

Pohst's original idea: in a graphical scheme C we can recognize patterns, corresponding to bounded factors of F_C.

$i\,\overset{j}{\boxed{+}} \leq 1$ since it corresponds to $(1-u) \leq 1$ for $u \in [0,1]$.

$i\,\overset{j}{\boxed{+}}\overset{j'}{\boxed{-}} \leq 1$ since it corresponds to $(1-u)(1+uv) \leq 1$ for $u, v \in [0,1]$.

$\begin{array}{c|c|} & j \\ i & - \\ i' & + \end{array} \leq 1$ for the very same reason.

Estimates and patterns of graphical schemes

Pohst's original idea: in a graphical scheme C we can recognize patterns, corresponding to bounded factors of F_C.

- $\begin{array}{c|c|} & j \\ \cline{2-2} i & + \\ \cline{2-2} \end{array} \leq 1$ since it corresponds to $(1-u) \leq 1$ for $u \in [0,1]$.
- $\begin{array}{c|c|c|} & j & j' \\ \cline{2-3} i & + & - \\ \cline{2-3} \end{array} \leq 1$ since it corresponds to $(1-u)(1+uv) \leq 1$ for $u, v \in [0,1]$.
- $\begin{array}{c|c|} & j \\ \cline{2-2} i & - \\ \cline{2-2} i' & + \\ \cline{2-2} \end{array} \leq 1$ for the very same reason.
- $\begin{array}{c|c|c|} & j & j' \\ \cline{2-3} i & - & + \\ \cline{2-3} i' & + & - \\ \cline{2-3} \end{array} \leq 1$ since $(1-u)(1+uv)(1+uw)(1-uvw) \leq 1$ for $u, v, w \in [0,1]$.
- $\begin{array}{c|c|c|} & j & j+1 \\ \cline{2-3} i & - & + \\ \cline{2-3} i+1 & \multicolumn{1}{c}{} & \multicolumn{1}{|c|}{-} \\ \cline{3-3} \end{array} \leq 2$ since it is nothing but a consequence of $Q_2(x_1, x_2) \leq 2$, which we already know.

To cover the scheme with patterns = to give an upper bound to F_C.

Configuration with negative signs

Theorem

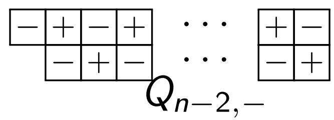
Let $Q_{n,-}$ be the configuration of Q_n defined by signs $(-1, -1, \ldots, -1)$.
Then $Q_{n,-}(x_1, \ldots, x_n) \leq 2^{\lfloor (n+1)/2 \rfloor}$.

Configuration with negative signs

Theorem

Let $Q_{n,-}$ be the configuration of Q_n defined by signs $(-1, -1, \dots, -1)$.
Then $Q_{n,-}(x_1, \dots, x_n) \leq 2^{\lfloor (n+1)/2 \rfloor}$.

Proof: If $n = 1$ or $n = 2$ the claim is trivial. Assume $n \geq 3$ is odd and the claim is true for every dimension less than n. The configuration is represented by the scheme

$Q_{n,-} =$ 

where $Q_{n-2,-} \leq 2^{\lfloor (n-1)/2 \rfloor}$ by hypothesis.

Configuration with negative signs

Theorem

Let $Q_{n,-}$ be the configuration of Q_n defined by signs $(-1,-1,\dots,-1)$. Then $Q_{n,-}(x_1,\dots,x_n) \leq 2^{\lfloor (n+1)/2 \rfloor}$.

Proof: If $n=1$ or $n=2$ the claim is trivial. Assume $n \geq 3$ is odd and the claim is true for every dimension less than n. The configuration is represented by the scheme

$$Q_{n,-} = \begin{array}{|c|c|c|c|cc|c|c|} \hline - & + & - & + & \cdots & & + & - \\ \hline & - & + & - & \cdots & & - & + \\ \hline \end{array}_{\;Q_{n-2,-}}$$

where $Q_{n-2,-} \leq 2^{\lfloor (n-1)/2 \rfloor}$ by hypothesis. In the first two rows we have one triangle $\begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & - & + \\ \hline 2 & & - \\ \hline \end{array} \leq 2$, we have $\lfloor \frac{n-2}{2} \rfloor$ squares $\begin{array}{c|c|c|} & j & j' \\ \hline 1 & - & + \\ \hline 2 & + & - \\ \hline \end{array} \leq 1$ and one final vertical segment $\begin{array}{c|c|} & n \\ \hline 1 & - \\ \hline 2 & + \\ \hline \end{array} \leq 1$.

Configuration with negative signs

Theorem

Let $Q_{n,-}$ be the configuration of Q_n defined by signs $(-1, -1, \ldots, -1)$. Then $Q_{n,-}(x_1, \ldots, x_n) \leq 2^{\lfloor (n+1)/2 \rfloor}$.

Proof: If $n = 1$ or $n = 2$ the claim is trivial. Assume $n \geq 3$ is odd and the claim is true for every dimension less than n. The configuration is represented by the scheme

$$Q_{n,-} = \begin{array}{|c|c|c|c|c|c|c|}\hline - & + & - & + & \cdots & + & - \\ \hline & - & + & - & \cdots & - & + \\ \hline \end{array}\quad \underbrace{}_{Q_{n-2,-}}$$

where $Q_{n-2,-} \leq 2^{\lfloor (n-1)/2 \rfloor}$ by hypothesis. In the first two rows we have one triangle $\begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & - & + \\ \hline 2 & & - \\ \hline \end{array} \leq 2$, we have $\lfloor \frac{n-2}{2} \rfloor$ squares $\begin{array}{c|c|c|} & j & j' \\ \hline 1 & - & + \\ \hline 2 & + & - \\ \hline \end{array} \leq 1$ and one final vertical segment $\begin{array}{c|c|} & n \\ \hline 1 & - \\ \hline 2 & + \\ \hline \end{array} \leq 1$. The contribution of the first two rows is then ≤ 2 and so

$$Q_{n,-} \leq 2 \cdot 2^{\lfloor (n-1)/2 \rfloor} = 2^{\lfloor \frac{n+1}{2} \rfloor}.$$

For $n \geq 4$ even the proof is completely similar.

Transforming patterns

If C, C' are graphical schemes of dimension n, we say $C \leq C'$ if $F_C \leq F_{C'}$.
New idea: instead of just detecting patterns, we replace them with other patterns and this corresponds to an estimate.

Transforming patterns

If C, C' are graphical schemes of dimension n, we say $C \leq C'$ if $F_C \leq F_{C'}$.
New idea: instead of just detecting patterns, we replace them with other patterns and this corresponds to an estimate.

P) $\ {}_i\overset{j}{\boxed{+}} \leq {}_i\overset{j}{\boxed{-}}$ since $(1-u) \leq (1+u)$ for $u \in [0,1]$.

H) $\ {}_i\overset{j\ \ j'}{\boxed{+}\boxed{-}} \leq {}_i\overset{j\ \ j'}{\boxed{-}\boxed{+}}$ since $(1-u)(1+uv) \leq (1+u)(1-uv)$ for $u, v \in [0,1]$.

Transforming patterns

If C, C' are graphical schemes of dimension n, we say $C \leq C'$ if $F_C \leq F_{C'}$.
New idea: instead of just detecting patterns, we replace them with other patterns and this corresponds to an estimate.

P) $\begin{array}{c|c|} & j \\ \hline i & + \\ \hline \end{array} \leq \begin{array}{c|c|} & j \\ \hline i & - \\ \hline \end{array}$ since $(1-u) \leq (1+u)$ for $u \in [0,1]$.

H) $\begin{array}{c|c|c|} & j & j' \\ \hline i & + & - \\ \hline \end{array} \leq \begin{array}{c|c|c|} & j & j' \\ \hline i & - & + \\ \hline \end{array}$ since $(1-u)(1+uv) \leq (1+u)(1-uv)$ for $u, v \in [0,1]$.

V) $\begin{array}{c|c|} & j \\ \hline i & - \\ \hline i' & + \\ \hline \end{array} \leq \begin{array}{c|c|} & j \\ \hline i & + \\ \hline i' & - \\ \hline \end{array}$ for the very same reason.

S) $\begin{array}{c|c|c|} & j & j' \\ \hline i & - & + \\ \hline i' & + & - \\ \hline \end{array} \leq \begin{array}{c|c|c|} & j & j' \\ \hline i & + & - \\ \hline i' & - & + \\ \hline \end{array}$ since for $u, v, w \in [0,1]$ we have

$$(1-u)(1+uv)(1+uw)(1-uvw) \leq (1+u)(1-uv)(1-uw)(1+uvw).$$

Every replacement is a move on C and produces a new scheme C' and an estimate $C \leq C'$.

The theorem

Theorem (B.-Molteni, 2021)

Let C be a configuration of Q_n. There is a list $\mathcal{L}$ of moves P, H, V, S which transforms C into the configuration $Q_{n,-}$ defined by negative signs.

Corollary

$$\max_{(x_1,\dots,x_n)\in[-1,1]^n} Q_n(x_1,\dots,x_n) = 2^{\lfloor \frac{n+1}{2} \rfloor}.$$

The theorem

Theorem (B.-Molteni, 2021)

Let C be a configuration of Q_n. There is a list $\mathcal{L}$ of moves P, H, V, S which transforms C into the configuration $Q_{n,-}$ defined by negative signs.

Corollary

$$\max_{(x_1,\dots,x_n)\in[-1,1]^n} Q_n(x_1,\dots,x_n) = 2^{\lfloor \frac{n+1}{2} \rfloor}.$$

* The list $\mathcal{L}$ is created by induction on the dimension n, moving through the columns.
* There is a precise algorithm that explains how to construct the list (the difficult part of the theorem is proving it always works).
* Sometimes moves of old columns are removed by new moves which partly overlap with the old ones.

An instance of the algorithm

We want to apply the algorithm to the following configuration of dimension 6.

$+$	$-$	$-$	$-$	$+$	$+$
	$-$	$-$	$-$	$+$	$+$
		$+$	$+$	$-$	$-$
			$+$	$-$	$-$
				$-$	$-$
					$+$

An instance of the algorithm

We want to apply the algorithm to the following configuration of dimension 6.

$+$	$-$	$-$	$-$	$+$	$+$
	$-$	$-$	$-$	$+$	$+$
		$+$	$+$	$-$	$-$
			$+$	$-$	$-$
				$-$	$-$
					$+$

$$\mathcal{L} = \{P[1;1]\}.$$

An instance of the algorithm

We want to apply the algorithm to the following configuration of dimension 6.

$+$	$-$	$-$	$-$	$+$	$+$
	$-$	$-$	$-$	$+$	$+$
		$+$	$+$	$-$	$-$
			$+$	$-$	$-$
				$-$	$-$
					$+$

$$\mathcal{L} = \{H[1;1,2]\}.$$

An instance of the algorithm

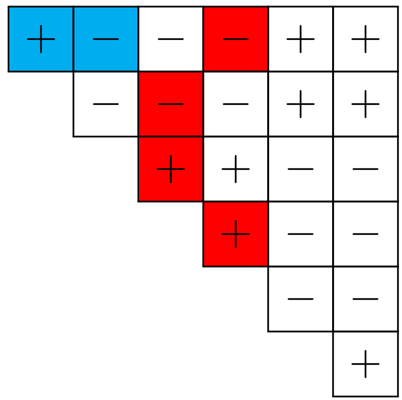
We want to apply the algorithm to the following configuration of dimension 6.

$+$	$-$	$-$	$-$	$+$	$+$
	$-$	$-$	$-$	$+$	$+$
		$+$	$+$	$-$	$-$
			$+$	$-$	$-$
				$-$	$-$
					$+$

$$\mathcal{L} = \{H[1;1,2], V[2,3;3]\}.$$

An instance of the algorithm

We want to apply the algorithm to the following configuration of dimension 6.



$$\mathcal{L} = \{H[1;1,2], V[2,3;3], V[1,4;4]\}.$$

An instance of the algorithm

We want to apply the algorithm to the following configuration of dimension 6.

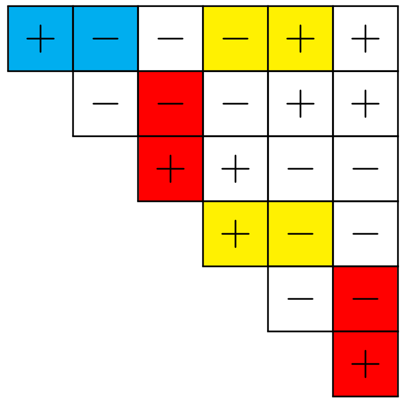
$+$	$-$	$-$	$-$	$+$	$+$
	$-$	$-$	$-$	$+$	$+$
		$+$	$+$	$-$	$-$
			$+$	$-$	$-$
				$-$	$-$
					$+$

$$\mathcal{L} = \{H[1;1,2], V[2,3;3], S[1,4;4,5]\}.$$

.

An instance of the algorithm

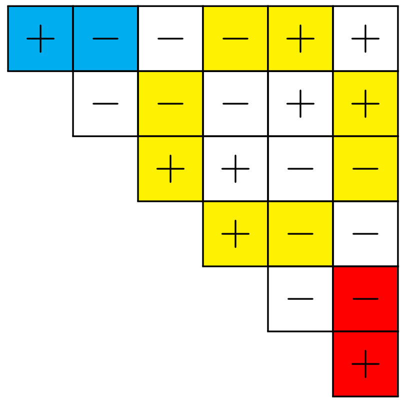
We want to apply the algorithm to the following configuration of dimension 6.



$$\mathcal{L} = \{H[1;1,2], V[2,3;3], S[1,4;4,5], V[5,6;6]\}.$$

An instance of the algorithm

We want to apply the algorithm to the following configuration of dimension 6.



$$\mathcal{L} = \{H[1;1,2], S[2,3;3,6], S[1,4;4,5], V[5,6;6]\}.$$

The case with 1 complex embedding

$$P_{n+1} := \prod_{1\leq i<j\leq n+1} \left|1 - \frac{\varepsilon_i}{\varepsilon_j}\right|.$$

FIRST PROBLEM: There is a couple of complex conjugated ε_k and ε_{k+1}: there are different changes of variables depending on the position of k.

$$\varepsilon_k = r_k e^{i\theta}, \quad g := \cos\theta, x_i := \begin{cases} \frac{\varepsilon_i}{\varepsilon_{i+1}} & i \neq k-1, k \\ \frac{\varepsilon_{k-1}}{r_k} & i = k-1, \\ \frac{r_k}{\varepsilon_{k+1}} & i = k \end{cases}$$

This results in several functions arising from the same P_{n+1}.
E.g: Signature $(3,1)$, ε_4 and ε_5 conjugated

$$\begin{array}{cccc} (1-x_1) & (1-x_1x_2) & (1-2x_1x_2x_3g+(x_1x_2x_3)^2) & \\ & (1-x_2) & (1-2x_2x_3g+(x_2x_3)^2) & \\ & & (1-2x_3g+x_3^2) & 2\sqrt{1-g^2} \end{array}$$

The case with 1 complex embedding

$$P_{n+1} := \prod_{1 \le i < j \le n+1} \left| 1 - \frac{\varepsilon_i}{\varepsilon_j} \right| .$$

FIRST PROBLEM: There is a couple of complex conjugated ε_k and ε_{k+1}: there are different changes of variables depending on the position of k.

$$\varepsilon_k = r_k e^{i\theta}, \quad g := \cos\theta, x_i := \begin{cases} \frac{\varepsilon_i}{\varepsilon_{i+1}} & i \neq k-1, k \\ \frac{\varepsilon_{k-1}}{r_k} & i = k-1, \\ \frac{r_k}{\varepsilon_{k+1}} & i = k \end{cases}$$

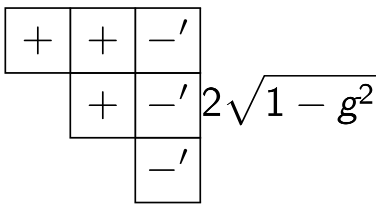
This results in several functions arising from the same P_{n+1}.
E.g: Signature $(3,1)$, ε_3 and ε_4 conjugated

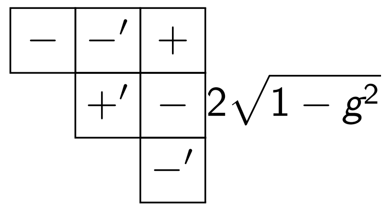
$$(1 - x_1) \quad (1 - 2x_1x_2g + (x_1x_2)^2) \qquad (1 - x_1x_2x_3)$$
$$(1 - 2x_2g + x_2^2) \qquad (1 - x_2x_3)$$
$$(1 - 2x_3g + x_3^2) \quad 2\sqrt{1 - g^2}$$

Problems with graphical schemes

Fortunately, there is symmetry between orderings: so we only have to check 2 orderings for $(3,1)$, 3 orderings for $(4,1)$ and $(5,1)$ and 4 orderings for $(6,1)$.

We can use again graphical schemes and configurations, with a new notation for the terms containing g.





Problems with graphical schemes

Fortunately, there is symmetry between orderings: so we only have to check 2 orderings for $(3,1)$, 3 orderings for $(4,1)$ and $(5,1)$ and 4 orderings for $(6,1)$.

We can use again graphical schemes and configurations, with a new notation for the terms containing g.

$$\begin{array}{|c|c|c|}\hline + & + & -' \\ \hline \multicolumn{1}{c|}{} & + & -' \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & -' \\ \cline{3-3}\end{array} 2\sqrt{1-g^2} \qquad \begin{array}{|c|c|c|}\hline - & -' & + \\ \hline \multicolumn{1}{c|}{} & +' & - \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & -' \\ \cline{3-3}\end{array} 2\sqrt{1-g^2}$$

SECOND PROBLEM: Many of the previous moves are no longer available if they involve the new, complicated factors.

E.G: no longer true that $\begin{array}{|c|c|}\hline + & -' \\ \hline\end{array} \leq \begin{array}{|c|c|}\hline - & +' \\ \hline\end{array}$ or $\begin{array}{|c|c|}\hline -' & + \\ \hline +' & - \\ \hline\end{array} \leq \begin{array}{|c|c|}\hline +' & - \\ \hline -' & + \\ \hline\end{array}$ but the move $\begin{array}{|c|}\hline -' \\ \hline +' \\ \hline\end{array} \leq \begin{array}{|c|}\hline +' \\ \hline -' \\ \hline\end{array}$ is still available.

The procedure for signature $(3, 1)$

* For the two orderings of $(3, 1)$, graphical schemes are not needed: partial derivatives are enough.
* We study consecutive resultants of partial derivatives with MAGMA and PARI/GP and obtain extremal points.
* The procedure is repeated when looking for critical points in the boundary.

The procedure for signature $(3,1)$

- For the two orderings of $(3,1)$, graphical schemes are not needed: partial derivatives are enough.
- We study consecutive resultants of partial derivatives with MAGMA and PARI/GP and obtain extremal points.
- The procedure is repeated when looking for critical points in the boundary.

$$\begin{array}{cccc} (1-x_1) & (1-x_1x_2) & (1-2x_1x_2x_3g+(x_1x_2x_3)^2) & \\ & (1-x_2) & (1-2x_2x_3g+(x_2x_3)^2) & \\ & & (1-2x_3g+x_3^2) & 2\sqrt{1-g^2} \end{array}$$

Maximum point: $(x_1,x_2,x_3,g)=\left(\frac{-1}{\sqrt{7}},-1,1,\frac{-1}{2\sqrt{7}}\right)$.

Maximum value: $16M=16.6965\ldots$ where $M=3^{15/2}/(4\cdot 7^{7/2})$.

Unfortunately, a direct procedure like this becomes unsustainable for the successive signatures.

The procedure for $(4,1), (5,1)$, $(6,1)$ and $(7,1)$

* For each of the 3 possible orderings (resp. 3, 4, 4) we consider the 16 (resp. 32, 64, 128) configurations available and the corresponding graphical schemes.
* Some of these schemes are estimated similarly to Pohst's procedure.
* Some of these patterns are complicated and estimated with the MAGMA/PARI procedure invoked before. Instead of 4 patterns, we have more than 100 to consider.
* Other schemes are transformed in schemes that we already know how to bound thanks to moves similar to the ones of totally real case.

Results on the orderings

Table of upper bounds for every configuration in every ordering.

(r_1, r_2) / ordering	(3,1)	(4,1)	(5,1)	(6,1)	(7,1)
1st	16M	32	32M	64M	155.1
2nd	16M	32M	54M	79.42	190.2
3rd		34.89	65.81	83.49	201.4
4th				83.90	233.1

Results on the orderings

Table of upper bounds for every configuration in every ordering.

ordering \ (r_1, r_2)	(3,1)	(4,1)	(5,1)	(6,1)	(7,1)
1st	16M	32	32M	64M	155.1
2nd	16M	32M	54M	79.42	190.2
3rd		34.89	65.81	83.49	201.4
4th				83.90	233.1

* The upper bounds for $(3, 1)$ and the ones in the first ordering for $(4, 1)$ and $(5, 1)$ are the best possible, since they are attained at specific values.
* Just like for $(3, 1)$, the true upper bounds should not depend on the ordering. They would be 32 for $(4, 1)$, $32M$ for $(5, 1)$, 64 for $(6, 1)$ and $64M$ for $(7, 1)$.

Results on the orderings

Table of upper bounds for every configuration in every ordering.

ordering \ (r_1, r_2)	(3,1)	(4,1)	(5,1)	(6,1)	(7,1)
1st	16M	32	32M	64M	155.1
2nd	16M	32M	54M	79.42	190.2
3rd		34.89	65.81	83.49	201.4
4th				83.90	233.1

* The upper bounds for $(3,1)$ and the ones in the first ordering for $(4,1)$ and $(5,1)$ are the best possible, since they are attained at specific values.
* Just like for $(3,1)$, the true upper bounds should not depend on the ordering. They would be 32 for $(4,1)$, $32M$ for $(5,1)$, 64 for $(6,1)$ and $64M$ for $(7,1)$.

Theorem (B.-Molteni,2022)

The fields K_1 and K_2 with signature $(6,1)$ and smallest regulator are the ones with smallest discriminant -65106259 and -68494627.
We have $R_{K_1} = 7.135\ldots$ and $R_{K_2} = 7.38\ldots$