Hecke operators and measures on $\mathbb{Z}_p$

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Let $\mathbb{Q}_p \subset L$ complete with discrete valuation group.
$L \supset \mathcal{O} \supset \mathfrak{m} = \pi\mathcal{O},\ |\pi| = e^{-1}$.

Let G be a compact group with ***p*-adic linear structure**:

$$\gamma : \mathbb{Z}_p \xrightarrow{\sim} G, \qquad \gamma(1) = g$$

A **measure** on G is a bounded linear functional on G.

Identifications

$$\begin{array}{ccccc} \mathrm{Meas}(G;\mathcal{O}) & \simeq & \mathcal{O}[[G]] & \simeq & \mathcal{O}[[T]] \\ \delta_g & \leftarrow & g & \to & 1+T \end{array}$$

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Let $\mu \in \mathrm{Meas}(G; \mathcal{O})$.
For all $n \geq 0$ set

$$m_n = \int_{\mathbb{Z}_p} x^n \, d\mu, \qquad m_n^\times = \int_{\mathbb{Z}_p^\times} x^n \, d\mu$$

Since

$$m_n(\mu) = m_n^\times(\mu) + \int_{p\mathbb{Z}_p} x^n \, d\mu$$

there are congruences

$$\boxed{m_{n'} \equiv m_n \bmod p^{\min\{r,n\}}}$$

se $1 \leq n < n'$ e $n' \equiv n \bmod p^{r-1}(p-1)$.

f: a weight k modular form for $\Gamma_0(N)$, $(p, N) = 1$.
E: an elliptic curve with CM in K with ordinary reduction $\widetilde{E}$ modulo p.
$x \in X_0(N)$: the point corresponding to E.
$\mathcal{M}$: the moduli space of formal deformations of $\widetilde{E} \otimes \overline{\mathbb{F}}_p$ to complete rings $\mathcal{O}$ finite over the ring of Witt vectors. It supports a universal elliptic curve $\mathcal{E}$

Concretely $\mathcal{M}$ is identified the formal neighborhood of x in $X_0(N)$ and $\mathcal{E}$ is the restriction of the universal family of elliptic curves.

Serre-Tate theory: There are canonical isomorphisms

- $\mathcal{M} = \mathrm{Spf}(\mathcal{R}) = \mathrm{Spf}(\mathcal{O}[[q-1]])$, q "local parameter",
- $\phi : \widehat{\mathcal{E}} \xrightarrow{\sim} \widehat{\mathbb{G}}_m$ "trivialization".

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So, we get a canonical power series expansion

$$f(\mathcal{E}, i_N, \phi) = F_f[[T] = \sum_{n \geq 0} a_{n,f} T^n \in \mathcal{O}[[q-1]] \quad (T = q-1).$$

The goal is to compute the action on the Hecke operator T_p on this expansion, namely compute the expansion of

$$\mathrm{T}_p(f)(\mathcal{E}, i_N, \phi) = \frac{1}{p} \sum_C f(\mathcal{E}/C, i_{N,C}, \phi_C)$$

where the sum is extended over the finite flat subgroups of $\mathcal{E}$ of rank p.

$\pi_C : \mathcal{E} \longrightarrow \mathcal{E}/C$ quotient map (of rank p)

- $C = H = \phi^{-1}(\mu_p)$ (canonical subgroup) connected
  $\Longrightarrow \pi_H$ lifts Frobenius, π_H^t étale
  $\Longrightarrow \phi_H = \phi \circ \pi_H^t$
- C étale (π_C^t lifts Frobenius)
  $\Longrightarrow \pi_C$ restricted to $\widehat{\mathcal{E}}$ is iso.
  $\Longrightarrow \phi_C = \phi \circ \pi_C^{-1}$

Extra-hypothesis: $\widetilde{E}$ is defined over $\mathbb{F}_p$.

$\Longrightarrow$ Frobenius is an endomorphism of $\widetilde{E}$ and so all quotients $\mathcal{E}/C$ are deformations of $\widetilde{E}$.
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Then one gets

- $q(\mathcal{E}/H) = q(\mathcal{E})^p$,
- $q(\mathcal{E}/C)^p = q(\mathcal{E})$ if $C \neq H$,

and eventually

$$\mathrm{T}_p f(\mathcal{E}, i_N, \phi) = \frac{1}{p}\left(\sum_{\zeta^p=1}\sum_{n\geq 0} a_{n,f}(\zeta q^{\frac{1}{p}} - 1)^n\right) + p^{k-1}\sum_{n\geq 0} a_{n,f}(q^p - 1)^n$$

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$$\begin{tikzcd}
p\mathbb{Z}_p & \mathbb{Z}_p \arrow[l, "p"'] \arrow[r, "p^{-1}"] & p^{-1}\mathbb{Z}_p \\
& \mathbb{Z}_p \arrow[ul, "\gamma_1"] \arrow[u, "\gamma_0"] \arrow[ur, "\gamma_{-1}"'] &
\end{tikzcd}$$

where $\gamma_j(1) = p^j$.

Given a measure μ let

$$\mathrm{T}_{p,k}(\mu) = i_*^{-1} {p^{-1}}_*(\mu_{|p\mathbb{Z}_p}) + p^{k-1} i_* p_*(\mu).$$

Proposition

If $\mu(f)$ is the measure associated to a modular form f as above, then

$$\mu(\mathrm{T}_p f) = \mathrm{T}_{p,k}\mu(f).$$

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Suppose $\mathrm{T}_p f = \lambda f$ so that $\mathrm{T}_{p,k}\mu_f = \lambda\mu_f$.

Then we have integration formulae ($U_n = p^n\mathbb{Z}_p^\times$):

- $\lambda \int_{\mathbb{Z}_p^\times} h(z)\, d\mu(z) = \int_{U_1} h(p^{-1}z)\, d\mu(z)$;
- $\lambda \int_{U_n} h(z)\, d\mu(z) = \int_{U_{n+1}} h(p^{-1}z)\, d\mu(z) + \int_{U_{n-1}} h(pz)\, d\mu(z)$.

Eventually

$$m_n(\mu) = \frac{1}{1 - \lambda p^n + p^{2n+k-1}} m_n^\times.$$

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Theorem

Suppose $\mathrm{T}_{p,k}\mu = \lambda\mu$ *with* $\lambda \in \mathbb{Z}_p$. *If* $1 \leq n < n'$ *with* $n' \equiv n \bmod p^{r-1}(p-1)$ *then*

$$m_{n'} \equiv m_n \bmod p^s\mathbb{Z}_p$$

where

$$s = \begin{cases} \min\{r, t+n, 2n+k-1\} & se\ \lambda \in p^t\mathbb{Z}_p^\times, \\ \min\{r, 2n+k-1\} & se\ \lambda = 0. \end{cases}$$

Note: the moments $m_n(\mu_f)$ can be computed applying certain non-holomorphic or p-adic differential operators to f and evaluating the result at the relevant CM point.