A curiosity about $(-1)^{[e]} + (-1)^{[2e]} + \cdots + (-1)^{[Ne]}$

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Let α be an irrational real number; the behavior of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$$

depends on the continued fraction expansion of $\alpha/2$. Since the continued fraction expansion of $\sqrt{2}/2$ has bounded partial quotients,

$$S_N(\sqrt{2}) = O(\log(N))$$

and this bound is best possible. The partial quotients of the continued fraction expansion of e grow slowly and thus

$$S_N(2e) = O\left(\frac{\log(N)^2}{\log\log(N)^2}\right),$$

again best possible. The partial quotients of the continued fraction expansion of $e/2$ behave similarly as those of e. Surprisingly enough

$$S_N(e) = O\left(\frac{\log(N)}{\log\log(N)}\right).$$