Finitely generated abelian groups of units

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Torino, 24-25 October 2019

Fuchs' questions

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> Characterize the groups which are the (abelian) groups of all units in a commutative and associative ring with identity.

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Partial approaches

- to restrict the class of rings
- to restrict the class of groups
- to restrict both

Units of number rings

Theorem (Dirichlet (1846))
Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Let $[K : \mathbb{Q}] = r + 2s$ (here r is the number of real embeddings of K in $\bar{\mathbb{Q}}$ and $2s$ the number of non-real embeddings). Then

$$\mathcal{O}_K^* \cong T \times \mathbb{Z}^{r+s-1}$$

where T is the (cyclic) group of the roots of unity contained in K.

Units in group rings

Let R be a ring and let G be a group. The group ring RG is defined by

$$RG = \{\sum_{g \in G} \lambda_g g \mid \lambda_g \in R \text{ and } \lambda_g = 0 \text{ for almost all } g\}.$$

Theorem (Higman 1940) Let G be a finite abelian group of order n. Then

$$(\mathbb{Z}G)^* \cong \pm G \times \mathbb{Z}^{r_G}$$

where $r_G = \frac{1}{2}(n + 1 + c_2 - 2l)$, with
$c_d = \#\{\text{cyclic subgroups of order d of } G\}$ and $l = \sum_{d|n} c_n$.

More recently

- Pearson and Schneider (1970):

  Classification of the realizable cyclic groups.
- Chebolu and Lockridge (2015):

  Classification of the realizable indecomposable abelian groups.
- idc, Dvornicich (2018)
  - Classification of the finite abelian groups which can be realized in the class of the integral domains, of the torsion-free rings and of the reduced rings.
  - necessary conditions for a f. ab. group to be realizable;
  - infinite new families of realizable/non-realizable finite abelian groups.
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Finitely generated abelian groups

Fuchs' question for finitely generated abelian groups

A ring with 1, A^* group of units of A. Assume that A^* is finitely generated and abelian

$$A^* \cong (A^*)_{tors} \times \mathbb{Z}^{r_A}$$

Problem: what groups arise?

- T finite abelian group: $\exists\, A \in \mathcal{C}$ such that $(A^*)_{tors} \cong T$?

- If $(A^*)_{tors} \cong T$ what can we say on $r_A = \mathrm{rank}(A^*)$?

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The problem

Reduction: Let $A_0(=\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z})$ be the fundamental subring of A and consider the ring $A_0[(A^*)_{tors}]$. Then

$$(A^*)_{tors} = (A_0[(A^*)_{tors}])^*_{tors}$$

and

$$r_A \geq r_{A_0[(A^*)_{tors}]}.$$

So, up to changing $A \longleftrightarrow A_0[(A^*)_{tors}]$, we can restrict to study: commutative rings which are finitely gen. and integral over A_0.

Integral domains

Theorem (idc 2019)

The finitely generated abelian groups that occur as groups of units of an integral domain are:
i) $\mathrm{char}(A) = p$*: all groups of the form* $\mathbb{F}_{p^n}^* \times \mathbb{Z}^r$ *with* $n \geq 1$ *and* $r \geq 0$*;*
ii) $\mathrm{char}(A) = 0$*: all groups of the form* $C_{2n} \times \mathbb{Z}^r$*, with* $n \geq 1$*,* $r \geq \frac{\phi(2n)}{2} - 1$*.*

Corollary

The finite abelian groups that occur as groups of units of an integral domain A are:
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Torsion-free rings

A is *torsion-free* if 0 is the only element of finite additive order. In this case, $\mathrm{char}(A) = 0$.
Example: If R is a torsion-free ring and G is a group, then RG is torsion-free.

Theorem (idc 2019)

Let T be a finite abelian group of even order. Then there exists an explicit constant $g(T)$ such that the following holds:

$$T \times \mathbb{Z}^r$$

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$$T \cong \prod_{\iota=1}^{s} C_{p_\iota^{a_\iota}} \times \prod_{i=1}^{\rho} C_{2^{\epsilon_i}} \times C_{2^\epsilon}^{\sigma}$$

where $s, \rho \geq 0$, $\sigma \geq 1$ and

- for all $\iota = 1, \ldots, s$ the p_ι's are odd prime numbers, not necessarily distinct, and $a_\iota \geq 1$;

- $\epsilon = \epsilon(T) \geq 1$ and $\epsilon_i > \epsilon$ for all $i = 1, \ldots, \rho$.

$$g(T) = \sum_{\iota=1}^{s} (\frac{\phi(2^\epsilon p_\iota^{a_\iota})}{2} - 1) + \sum_{i=1}^{\rho} (\frac{\phi(2^{\epsilon_i})}{2} - 1) + c(T)$$

where

$$c(T) = \begin{cases} (\sigma - s)(\frac{\phi(2^\epsilon)}{2} - 1) & \text{for } s < \sigma \text{ and } \epsilon > 1 \\ 0 & \text{for } s_0 \leq \sigma \leq s \text{ or } \epsilon = 1 \\ \left\lceil \frac{\phi(2^\epsilon)}{2} - 1 \right\rceil & \text{for } \sigma < s_0 \end{cases}$$

where $s_0 = \#\{p_1, \ldots, p_s\}$.

Sketch of the proof:

1. We can assume assume A of the form $\mathbb{Z}[(A^*)_{tors}]$.
2. $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is semisimple and is a "cyclotomic $\mathbb{Q}$-algebra", namely

$$\mathbb{Q} \otimes_{\mathbb{Z}} A \cong \prod_{i=1}^{t} \mathbb{Q}(\zeta_{n_i}).$$

Since A is torsion-free, we can reduce to study orders in its maximal order

$$\mathcal{M} = \prod_{i=1}^{t} \mathbb{Z}[\zeta_{n_i}].$$

3. If A is an order of $\mathcal{M}$ then $\operatorname{rank}(A^*) = \operatorname{rank}(\mathcal{M}^*)$.

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$(A^*)_{tors} \cong T \Longrightarrow \mathrm{rank}(A^*) \geq g(T)$.

$\forall\ r \geq g(T) \Longrightarrow \exists A$ such that $A^* \cong T \times \mathbb{Z}^r$.

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We say that $\mathcal{M} = \prod_{i=1}^{t} \mathbb{Z}[\zeta_{n_i}]$ is T-admissible if

- $T \leq \mathcal{M}^*$ (arithmetic conditions on the n_i's)
- $2^{\epsilon} \mid n_i$ for all i.

- If $A \subseteq \mathcal{M} = \prod_{i=1}^{t} \mathbb{Z}[\zeta_{n_i}]$ then $\mathcal{M}$ is T-admissible.
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- but $\mathcal{M}_{0,T} = \prod_{i=1}^{s} \mathbb{Z}[\zeta_{2^\epsilon p_i^{a_i}}] \times \prod_{\iota=1}^{\rho} \mathbb{Z}[\zeta_{2^{\epsilon_\iota}}] \times \mathbb{Z}[\zeta_{2^\epsilon}]^{\max\{\sigma - s, 0\}}$ is T-admissible and

$$\mathrm{rank}(\mathcal{M}_{0,T}^*) = \begin{cases} g(T) & \text{for } \sigma \geq s_0 \\ g(T) - \left\lceil \frac{\phi(2^\epsilon)}{2} - 1 \right\rceil & \text{for } \sigma < s_0. \end{cases}$$

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The case $\sigma < s_0$

In this case each order $\mathcal{O} \subseteq \mathcal{M}_{0,T}$ with $T \leq (\mathcal{O}^*)_{tors}$ contains too many elements of order 2, so $T \lneq (\mathcal{O}^*)_{tors}$.

Example: Let $T = C_2 \times C_p \times C_q$ where $p \neq q$ are odd primes. Then

$$\mathcal{M}_{0,T} = \mathbb{Z}[\zeta_p] \times \mathbb{Z}[\zeta_q]$$

and

$$(\mathcal{M}_{0,T}^*)_{tors} \cong C_2 \times T$$

If $T \leq (\mathcal{O}^*)_{tors}$, then all the p and q elements of $(\mathcal{M}_{0,T}^*)_{tors}$ belongs to T, so $\alpha = (\zeta_p, \zeta_q) \in \mathcal{O}$. Now

$$\mathcal{O} \supseteq \mathbb{Z}[\alpha] \cong \frac{\mathbb{Z}[x]}{(\Phi_p(x)\Phi_q(x))} \cong \mathbb{Z}[\zeta_p] \times \mathbb{Z}[\zeta_q] = \mathcal{M}_{0,T},$$

hence $\mathcal{O} = \mathcal{M}_{0,T}$ and $(\mathcal{O}^*)_{tors} \cong C_2 \times T$.

Construction of A with $A^* \cong T \times \mathbb{Z}^{g(T)}$

Let

$$\mathcal{M}_T = \begin{cases} \mathcal{M}_{0,T} & \text{for } \sigma \geq s_0 \\ \mathcal{M}_{0,T} \times \mathbb{Z}[\zeta_{2^\epsilon}] & \text{for } \sigma < s_0, \end{cases}$$

then $\mathrm{rank}(\mathcal{M}_T) = g(T)$ and $(\mathcal{M}_T^*)_{tors} \cong T \times C_{2^\epsilon}^{\cdots\cdots}$.

Claim: we can construct A inside $\mathcal{M}_T$.

Example: Let $T = C_2 \times C_p \times C_q$ where $p \neq q$ are odd primes.

$$\mathcal{M}_T = \mathcal{M}_{0,T} \times \mathbb{Z} = \mathbb{Z}[\zeta_p] \times \mathbb{Z}[\zeta_q] \times \mathbb{Z}$$

$$(\mathcal{M}_T^*)_{tors} \cong C_2^2 \times T$$

Let $\alpha = (\zeta_p, 1, 1), \beta = (1, \zeta_q, 1)$ and put $A = \mathbb{Z}[\alpha, \beta]$.

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Let $u = (u_1, u_2, u_3) \in A$ with $u^2 = 1$:

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Let $u = (u_1, u_2, u_3) \in A$ with $u^2 = 1$: then (u_1, u_3) is a unit of exponent 2 of the ring $\mathbb{Z}[(\zeta_p, 1)]$ and one can show that

$$(\mathbb{Z}[(\zeta_p, 1)]^*)_{tors} \cong C_{2p}$$

so $u_1 = u_3 = \pm 1$.

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Analogously, $u_2 = u_3 = \pm 1$, hence $u = (1, 1, 1)$ or $u = (-1, -1, -1)$.

Corollary (idc, RD)

The finite abelian groups which are the group of units of a torsion-free ring A, are all those of the form

$$C_2^a \times C_4^b \times C_3^c$$

where $a, b, c \in \mathbb{N}$, $a + b \geq 1$ and $a \geq 1$ if $c \geq 1$.
In particular, the possible values of $|A^*|$ are the integers $2^d 3^c$ with $d \geq 1$.

Reduced rings

Theorem (idc 2019)

The finitely generated abelian groups that occur as groups of units of a reduced ring are those of the form

$$\prod_{i=1}^{k} \mathbb{F}^{*}_{p_i^{n_i}} \times T \times \mathbb{Z}^g$$

where $k, n_1, \ldots, n_k$ *are positive integers,* $\{p_1, \ldots, p_k\}$ *are not necessarily distinct primes,* T *is any finite abelian group of even order and* $g \geq g(T)$.

The general case?

If A is any ring and $\mathfrak{N}$ denotes its radical, then we can try to study A^* via the exact sequence

$$1 \to 1 + \mathfrak{N} \to A^* \to (A/\mathfrak{N})^* \to 1$$

In two joint papers with R. Dvornicich we used this method to derive information on realizable finite abelian groups without restriction on the ring.

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Grazie per l'attenzione!