Integral Points on Surfaces - Some open problems and recent results

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The complex solutions $(x_1, \ldots, x_N) \in \mathbb{C}^N$ form an *algebraic variety*.

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YES:

$$(x, y, z) = (8866128975287528, 8778405442862239, 2736111468807040)$$

New equation:

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Replace 42 by 114: still unsolved.

All the above equations admit a Zariski-dense set of rational solutions.

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give rise to non-zero integers which can be written as sums of two cubes into two different ways. There is a Zariski-dense set of rational points.

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Applying this method to the taxi-cab surface, starting e.g. from the Ramanujan point $(1 : 12 : 9 : 10)$, one obtains infinitely many taxi-cab numbers.

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Let r, s_0, s_1 be the lines

$$r : \begin{cases} X = Z \\ Y = W \end{cases} \qquad s_0 : \begin{cases} X = -W \\ Y = Z \end{cases} \qquad s_1 : \begin{cases} X = W \\ Y = -Z \end{cases}$$

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Then $S(\mathbb{Q})$ is Zariski-dense. It is also dense in the usual topology, in the set of real points $S(\mathbb{R})$.

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Conjecture [Bombieri-Lang]. *The rational points on surfaces of general type, on any given number field, are not Zariski-dense.*

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A more general theorem of Faltings:

Theorem [Faltings]. *Given an abelian variety A and an algebraic subvariety $X \subset A$, all defined over number field κ, the set $X(\kappa)$ is contained in a finite union of translates of abelian subvarieties of A contained in X.*

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More generally, to algebraic varieties rationally dominating a variety which can be embedded in an abelian variety.

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$$\pi(X(\kappa')) \supset Y(\kappa).$$

Bombier-Lang Conjecture applies also to varieties admitting an étale cover dominating a variety of general type.

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Letting $X = \tilde{X} \setminus D$ the quasi projective variety obtained by removing the closed subvariety D, we denote by $X(\mathcal{O}_S)$ the set of S-integral points of $\tilde{X}$ with respect to D.
Up to enlarging S, the density of $X(\mathcal{O}_S)$ only depends on the abstract quasi projective variety X over κ, not on the compactification $\tilde{X}$ nor on its projective embedding $\tilde{X} \hookrightarrow \mathbb{P}_n$.

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Letting $K_{\tilde{X}}$ be a canonical divisor, suppose that the sum

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is a big divisor. Then $X(\mathcal{O}_S)$ is not Zariski-dense.

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$$u + v + w = 1$$

to be solved in S-units $u, v, w \in \mathcal{O}_S^*$.

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$$n = a10^h + b10^k,$$

where $0 < a \leq 9$, $0 < b \leq 9$, $0 \leq h < k$.

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$$n^2 = a^2 10^{2h} + 2ab10^{h+k} + b^2 10^{2k},$$

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where $0 < a \leq 9$, $0 < b \leq 9$, $0 \leq h < k$. In base ten it has only 2 non-zero digits. Since

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A way of applying Vojta's Theorem: given a quasi-projective variety X, find an étale cover $Y \to X$ and a morphism $Y \to G$ to a semi-abelian variety whose image is not an algebraic subgroup of G.

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A pair (P_1, P_2) of rational points in $E_1 \times E_2$ with $d(P_1) = d(P_2)$ lifts to an integral point of S.

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The statement amounts to the degeneracy of integral points on a surface S' with $q(S') = 3$.

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are not Zariski-dense in the plane.