Density related with groups of rational numbers

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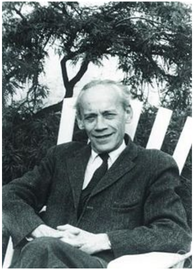
Artin Conjectures (AC)

Carolus Fridericus Gauß
(30 Apr 1777 – 23 Feb 1855)



Emil Artin
(Mar 3, 1898 – Dec 20, 1962)

Conjecture (Weak AC)

- $\forall a \in \mathbb{Q} \setminus \{-1\}, a \notin (\mathbb{Q})^2$
- $P_a := \{p \geq 3, \langle a \bmod p \rangle = \mathbb{F}_p^*\}$
- $\# P_a = \infty$

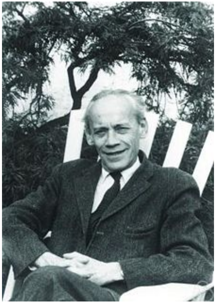
Conjecture (Strong AC)

- $\forall a \in \mathbb{Q}$, P_a has a density
  $$\delta_a = \lim_{x \to \infty} \frac{P_a \cap [1, x]}{\#\{p \leq x\}}$$
- $\delta_a = 0 \iff a = -1$ or $a \in \mathbb{Q}^2$

Note (Obvious Facts:)

- Strong AC $\implies$ Weak AC
- if $a \in \mathbb{Q}^2 \implies$
  - P_a has a density
  - $\delta_a = 0$ (trivial case of AC)

Strong Artin Conjecture
after Lehmer's correction



Emil Artin
(Mar 3, 1898 - Dec 20, 1962)

Conjecture (Strong AC(1927))

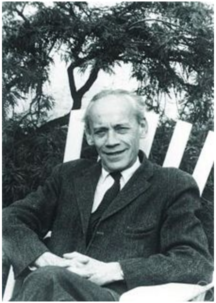
Let $a \in \mathbb{Q} \setminus \{0, 1, -1\}$. Then

$$P_a(x) \sim \delta_a \times \pi(x)$$

where $\delta_a = \delta'_a \times \delta''_a \times A$,

- $\delta''_a = \prod_{\ell | h} \frac{\ell(\ell-2)}{\ell(\ell-1)-1}$ and $h = \max\{k : a \in (\mathbb{Q}^*)^k\}$
- $\partial(a) := \mathrm{disc}(\mathbb{Q}(\sqrt{a}))$
- $\delta'_a = \begin{cases} 1 & \text{if } 2 \mid \partial(a) \\ 1 - \displaystyle\prod_{\ell | \partial(a)} \frac{-1}{(\ell, h)(\ell - 1) - 1} & \text{otherwise} \end{cases}$
- $A = \prod_{\ell} \left(1 - \frac{1}{\ell(\ell - 1)}\right) = 0.373955813619 20 \cdots$
  Artin constant
- $\delta_a = \delta'_a = \delta''_a = 0 \iff a \in (\mathbb{Q}^*)^2$

Some numerical tests for Artin Conjecture


Emil Artin


Derrick Henry Lehmer

Let

$$d_a = \#\{p \leq 2^{29} : \langle a \bmod p \rangle = \mathbb{F}_p^*\}/\pi(2^{29})$$

a	δ_a	d_a	a	δ_a	d_a
-15	0.37001	0.37005	2	0.37395	0.37397
-14	0.37395	0.37397	3	0.37395	0.37398
-13	0.37395	0.37395	5	0.39363	0.39365
-12	0.44875	0.44881	6	0.37395	0.37398
-11	0.37709	0.37736	7	0.37395	0.37401
-10	0.37395	0.37396	8	0.22437	0.22438
-9	0.37395	0.37395	10	0.37395	0.37395
-8	0.22437	0.22437	11	0.37395	0.37395
-7	0.38308	0.38304	12	0.37395	0.37403
-6	0.37395	0.37398	13	0.37636	0.37639
-5	0.37395	0.37392	14	0.37395	0.37395
-4	0.37395	0.37396	15	0.37395	0.37400
-3	0.44875	0.44874	17	0.37533	0.37537
-2	0.37395	0.37393	18	0.37395	0.37404

Christopher Hooley
(August 7, 1928 –)

Theorem (C. Hooley (1965))

Let $a \in \mathbb{Q} \setminus \{-1, 0, 1\}$ and assume GRH for $\mathbb{Q}[a^{1/m}]$, $m \in \mathbb{N}$. Then the Strong Artin Conjecture holds:

$$P_a(x) = \delta_a \frac{x}{\log x} + O\left(\frac{x \log\log x}{\log^2 x}\right).$$

Andrzej Schinzel
(April 5, 1937 –)

Theorem (A. Schinzel (1958))

The weak form of AC follows from Hypothesis H

Andrzej Schinzel
(April 5, 1937 –)

Conjecture (Schinzel's Hypothesis H)

Let $f_1, \dots, f_r \in \mathbb{Z}[x]$ be such that

- irreducible
- positive leading coefficients
- $\gcd(f_1(n) \cdots f_r(n) | n \in \mathbb{N}) = 1$ (*Bunyakovsky's property*)
  (i.e. $\forall \ell$ prime, $\exists n$: $\ell \nmid f_1(n) \cdots f_r(n)$)

$\Longrightarrow \exists \infty$–many $n \in \mathbb{N}$ s.t. $f_1(n), \dots, f_r(n)$ are all prime

Example

- $f_1 = x$; $f_2 = x + 2$: ∞–many *twin* primes
- $f_1 = x$; $f_2 = 2x + 1$: ∞–many *Sophie Germain* primes
- $x^2 + x + 2$ has NOT the *Bunyakovsky's property*

The near resolution of the WEAK AC


Rajiv Gupta


Ram Murty


Roger Heath–Brown

Lemma (Heath–Brown (1986))

$\exists \alpha \in \left(\frac{1}{4}, \frac{1}{2}\right]$ *s.t.*

$$\#\left\{p \leq x : \begin{array}{c} p \equiv u \bmod v \text{ and either } p = kq+1 \\ \text{or } p = kq_1q_2+1 \text{ with } q_i \geq \left[\frac{p-1}{k}\right]^{\alpha} \end{array}\right\} \gg \frac{x}{(\log x)^2}$$

as long as $k = 2, 4, 8$, $u \equiv 1 \bmod k$, $16 \mid V$ *and* $\gcd\left(\frac{u-1}{k}, v\right) = 1$

(Consequences)

- Given $q, r, s \in \mathbb{Z}$ multiplicatively independent, s.t. none of $q, r, s, -3qr, -3qs, -3rs, qrs$ is a square, $\exists a \in \{q, r, s\}$ with
$$P_a(x) := P_a \cap [1, x] \gg \frac{x}{(\log x)^2}$$
- $\exists g \in \{2, 3, 5\}$ s.t.
$$\#\{p \leq x : \ p > 5, \langle g \bmod p\rangle = \mathbb{F}_p^*\} \gg \frac{\pi(x)}{\log x}$$

The Artin near–primitive root Conjecture

après Lenstra, Moree, Murata, Wagstaff


Samuel Wagstaff


Leo Murata


Hendrik Lenstra


Pieter Moree

Definition (primes with a as a near–primitive root)

Let

$$P_{a,m} = \{p \geq 3 : [\mathbb{F}_p^* : \langle a \bmod p \rangle] = m\}$$

Theorem

Assume the GRH for $\mathbb{Q}[\sqrt[n]{a}]$, $n \in \mathbb{N}$. Then

$$P_{a,m}(x) := P_{a,m} \cap [1, x] = \delta_{a,m} \frac{x}{\log x} + O_{a,m}\left(\frac{x}{\log^2 x}\right)$$

where

$$\delta_{a,m} = \sum_{n \in \mathbb{N}} \frac{\mu(n)}{[\mathbb{Q}[e^{\frac{2\pi i}{mn}}, \sqrt[mn]{a}] : \mathbb{Q}]}.$$

For $k \in \mathbb{N}$ ℓ prime, $k_\ell := \ell^{v_\ell(k)}$. Write $a = \pm b^h$ with b not a power of a rational number and set $d = \mathrm{disc}(\mathbb{Q}(\sqrt{b}))$. Let

$$A(a,m) = \frac{(m,h)}{m^2} \times \prod_{\substack{\ell \mid m \\ h_\ell \mid m_\ell}} \left(1 + \frac{1}{\ell}\right) \times \prod_{\ell \nmid m} \left(1 - \frac{(\ell,h)}{\ell^2 - \ell}\right).$$

Then

$$\delta_{a,m} = A(a,m)\left(x_{a,m} + y_{a,m} \times \prod_{\ell \mid d, \ell \nmid 2m} \frac{-(\ell,h)}{\ell^2 - \ell - (\ell,h)}\right)$$

where, if $a > 0$, $x_{a,m} = 1$ and $y_{a,m} = \begin{cases} 1 & \text{if } \max\{2h_2, d_2\} \le m_2; \\ -1/3 & \text{if } \max\{2h_2, d_2\} = 2m_2 \ne 2; \\ -1 & \text{if } \max\{2h_2, d_2\} = 2m_2 = 2; \\ 0 & \text{if } \max\{2h_2, d_2\} > 2m_2, \end{cases}$

and if $a < 0$,

$(x_{a,m}, y_{a,m}) \in \{(1,1), (1,-1/3), (1,-1), (1,0), (1/3,-1/3), (1/2,0), (1/3,0)\}$

Hendrik Lenstra

Theorem (Lenstra - 1977)

Let $a \in \mathbb{Q}^* \setminus \{\pm 1\}$, $m \in \mathbb{N}$ and set $\partial(a) = \text{disc}\,\mathbb{Q}(\sqrt{a})$. Then $\delta_{a,m} = 0$ if and only if a and m satisfy one of the following (mutually exclusive) conditions holds:

1. $\partial(a) \mid m$ and m is odd
2. $a = u^{3\cdot 2^{t-1}}$, $\partial(-3u) \mid m$, $3 \nmid m$, $2^t \mid m$ $\quad \exists u \in \mathbb{Q}, t \in \mathbb{N}$
3. $a = -u^2$, $\quad \partial(2u) \mid 2m$, $m \equiv 2 \bmod 4$ $\quad \exists u \in \mathbb{Q}$
4. $a = -u^{3\cdot 2^t}$, $\partial(-3u) \mid m$, $3 \nmid m$, $2^{t+2} \mid m$ $\quad \exists u \in \mathbb{Q}, t \in \mathbb{N}$
5. $a = -u^6$, $\quad \partial(-6u) \mid m$, $3 \nmid m$, $m \equiv 4 \bmod 8$ $\quad \exists u \in \mathbb{Q}$.

Example

	1	2	3	4	5
(a, m)	$(5, 5)$	$(3^3, 4)$	$(-6^2, 6)$	$(-3^3, 4)$	$(-6^6, 4)$

Corollary (GRH: $\#P_{a,m} = \infty$ for (a, m) $\iff$ none above of the holds for (a, m))

Near–primitive root Conjecture for subgroups of $\mathbb{Q}^*$

Notations:

- $\Gamma \subset \mathbb{Q}^*$ finitely generated subgroup
- $r = \operatorname{rank} \Gamma$
- $m \in \mathbb{N}^+$
- $\sigma_\Gamma = \prod_{p : v_p(x)=0, \exists x \in \Gamma} p$
- $\forall p \nmid \sigma_\Gamma$

$$\Gamma_p = \{g (\operatorname{mod} p) : g \in \Gamma\} \subset \mathbb{F}_p^*$$

  is well defined
- $P_\Gamma := \#\{p \geq 3 : p \nmid \sigma_\Gamma, |\Gamma_p| = \frac{p-1}{m}\}$
- Γ_p generalizes the notion of $\langle a \bmod p \rangle$.
- if $\Gamma = \langle a \rangle$ has rank 1 then

$$P_{\langle a \rangle, m} = P_{a,m}$$

The higher rank Artin Quasi–primitive root Conjecture

joint work with Andrea Susa

Andrea Susa

FP

Theorem

Let $\Gamma \subset \mathbb{Q}^*$ has rank $r \geq 2$, let $m \in \mathbb{N}$ and assume GRH holds for $\mathbb{Q}(\zeta_k, \Gamma^{1/k})$ $(k \in \mathbb{N})$. Then, $\forall \epsilon > 0$ and $m \leq x^{\frac{r-1}{(r+1)(4r+2)} - \epsilon}$,

$$P_{\Gamma,m}(x) := P_{\Gamma,m} \cap [1, x] = \left(\delta_{\Gamma,m} + O\left(\frac{1}{\log^r x}\right)\right) \pi(x),$$

where

$$\delta_{\Gamma,m} = \sum_{k \geq 1} \frac{\mu(k)}{[\mathbb{Q}(\zeta_{mk}, \Gamma^{1/mk}) : \mathbb{Q}]}.$$

- $\Gamma^{1/k} := \{\sqrt[k]{t} : t \in \Gamma\} \subset \mathbb{R}$
- $\zeta_k := e^{2\pi i/k}$

How to compute $\delta_{\Gamma,m}$

$$\delta_{\Gamma,m} = \sum_{k\geq 1} \frac{\mu(k)}{[\mathbb{Q}(\zeta_{mk}, \Gamma^{1/mk}) : \mathbb{Q}]}$$

- $\Gamma^{1/k} := \{\sqrt[k]{t} : t \in \Gamma\} \subset \mathbb{R}$
- $\zeta_k := e^{2\pi i/k}$

Kummer Theory

- $[\mathbb{Q}(\zeta_n, \Gamma^{1/n}) : \mathbb{Q}] = [\mathbb{Q}(\zeta_n, \Gamma^{1/n}) : \mathbb{Q}(\zeta_n)][\mathbb{Q}(\zeta_n) : \mathbb{Q}]$
- $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$
- $[\mathbb{Q}(\zeta_n, \Gamma^{1/n}) : \mathbb{Q}(\zeta_n)] = \#\,\mathrm{Gal}(\mathbb{Q}(\zeta_n, \Gamma^{1/n})/\mathbb{Q}(\zeta_n))$
- $\mathrm{Gal}(\mathbb{Q}(\zeta_n, \Gamma^{1/n})/\mathbb{Q}(\zeta_n)) \cong (\mathbb{Q}(\zeta_n)^*)^n\Gamma/(\mathbb{Q}(\zeta_n)^*)^n$
- $\mathbb{Q}(\zeta_n)^{*n}\Gamma/\mathbb{Q}(\zeta_n)^{*n} \cong \left(\mathbb{Q}^{*n}\Gamma/\mathbb{Q}^{*n}\right) / \left(\mathbb{Q}(\zeta_n)^{*n} \cap \Gamma\mathbb{Q}^{*n}/\mathbb{Q}^{*n}\right)$

How to compute $\delta_{\Gamma,m}$

If we set

$$\Gamma(n) := {\mathbb{Q}^*}^n\Gamma/{\mathbb{Q}^*}^n \qquad \tilde{\Gamma}(n) := \left(\mathbb{Q}(\zeta_n)^{*n} \cap \Gamma{\mathbb{Q}^*}^n/{\mathbb{Q}^*}^n\right)$$

then

$$\delta_{\Gamma,m} = \sum_{k\geq 1} \frac{\mu(k)\#\tilde{\Gamma}(mk)}{\varphi(mk)\#\Gamma(mk)}$$

- $\#\Gamma(n)$ is a multiplicative function.
- $\tilde{\Gamma}(n) = \{\gamma \in \Gamma(n) : \gamma' \in \Gamma \cdot {\mathbb{Q}^*}^n \cap (\mathbb{Q}(\zeta_n)^*)^n\}$
  where for $\gamma \in \Gamma(n)$, $\gamma' \in \mathbb{Z}$ unique n–power free s.t. $\gamma = \gamma' \cdot {\mathbb{Q}^*}^n$
- $\#\tilde{\Gamma}(n) = 1$ if $2 \nmid n$
- $\tilde{\Gamma}(n) = \{\gamma \in \Gamma(n_2) : \gamma' \in \Gamma \cdot {\mathbb{Q}^*}^{n_2} \cap (\mathbb{Q}(\zeta_n)^*)^{n_2}\}$

Andrea Susa

FP

Theorem (2013)

Assume $\Gamma \subset \mathbb{Q}^+$, $r = \operatorname{rank}\Gamma \geq 2$ *and* $m \in \mathbb{N}$. $m_\ell := \ell^{v_\ell(m)}$

$$\delta_{\Gamma,m} = A_{\Gamma,m}\left(B_{\Gamma,m} - \frac{|\Gamma(m_2)|}{(2,m)|\Gamma(2m_2)|}B_{\Gamma,2m}\right)$$

where

$$A_{\Gamma,m} = \frac{1}{\varphi(m)|\Gamma(m)|} \times \prod_{\substack{\ell>2\\ \ell\nmid m}}\left(1 - \frac{1}{(\ell-1)|\Gamma(\ell)|}\right) \times \prod_{\substack{\ell>2\\ \ell\mid m}}\left(1 - \frac{|\Gamma(m_\ell)|}{\ell|\Gamma(\ell m_\ell)|}\right)$$

and

$$B_{\Gamma,k} = \sum_{\substack{\eta\mid\sigma_\Gamma\\ \eta^{k_2/2}\mathbb{Q}^{*k_2}\in\Gamma(k_2)\\ v_2(\partial(\eta))\leq k}} \prod_{\substack{\ell\mid\partial(\eta)\\ \ell\nmid k}} \frac{-1}{(\ell-1)|\Gamma(\ell)|-1}$$

Theorem (Susa, FP - 2013)

Let $\Gamma \subset \mathbb{Q}^+$ f.g., $m \in \mathbb{N}$. Then

$$\delta_{\Gamma,m} = 0$$

if one of the following holds:

1. $2 \nmid m$ *and* $\forall g \in \Gamma, \partial(g) \mid m$;
2. $2 \mid m$, $3 \nmid m$, $\Gamma(3) = \{1\}$ *and* $\exists \eta \mid \sigma_\Gamma$,
   - $\eta^{m_2-1}\mathbb{Q}^{*m_2} \in \Gamma(m_2)$
   - $\partial(-3\eta) \mid m$

Furthermore if $\Gamma \subset \mathbb{Q}^+$ and m satisfy ❶ or ❷ , then $P_{\Gamma,m}$ is finite

- If $\Gamma = \langle a \rangle$, conditions specialize to ❶ or ❷ of Lenstra
- If $2 \nmid m$, ❶ is also necessary for $\delta_{\Gamma,m} = 0$
- Hence, on GRH,
$$P_{\Gamma,2n+1} = \emptyset \iff \forall g \in \Gamma, \partial(g) \mid 2n+1$$
- If $2 \mid m$, expect that ❷ also necessary

The r–rank Artin near–primitive root Conjecture

joint work with Andam Mustafa Ali (general groups)

Andam Mustafa Ali

FP

Theorem (2019)

Assume $\Gamma \subset \mathbb{Q}^*$, $r = \operatorname{rank}\Gamma \geq 2$ *and* $m \in \mathbb{N}$. $m_\ell := \ell^{v_\ell(m)}$

$$\delta_{\Gamma,m} = A_{\Gamma,m}\left(B_{\Gamma,m} + C_{\Gamma,m} - \frac{|\Gamma(m_2)|}{(2,m)|\Gamma(2m_2)|}\left(B_{\Gamma,2m} + C_{\Gamma,2m}\right)\right)$$

where

$$A_{\Gamma,m} = \frac{1}{\varphi(m)|\Gamma(m)|} \times \prod_{\substack{\ell>2\\ \ell\nmid m}}\left(1 - \frac{1}{(\ell-1)|\Gamma(\ell)|}\right) \times \prod_{\substack{\ell>2\\ \ell\mid m}}\left(1 - \frac{|\Gamma(m_\ell)|}{\ell|\Gamma(\ell m_\ell)|}\right)$$

and

$$B_{\Gamma,k} = \sum_{\substack{\eta\mid\sigma_\Gamma\\ \eta^{k_2/2}\cdot\mathbb{Q}^{*k_2}\in\Gamma(k_2)\\ \partial(\eta)_2\mid k_2}} \prod_{\substack{\ell\mid\partial(\eta)\\ \ell\nmid k}} \frac{-1}{(\ell-1)|\Gamma(\ell)|-1} \text{ and } C_{\Gamma,k} = \sum_{\substack{\eta\in\Gamma(k_2)\\ \eta=-u^{k_2/2}\\ \partial(u)_2\mid 2k_2}} \prod_{\substack{\ell\mid\partial(u)\\ \ell\nmid k}} \frac{-1}{(\ell-1)|\Gamma(\ell)|-1}$$

Recall that

$$\tilde{\Gamma}(n) = \{\gamma \in \Gamma(n_2) : \gamma' \in \Gamma \cdot {\mathbb{Q}^*}^{n_2} \cap (\mathbb{Q}(\zeta_n)^*)^{n_2}\}$$

Lemma

Let $\Gamma \subset \mathbb{Q}^*$ and $n \in \mathbb{N}$. Set

$$\tilde{\Gamma}(m)^+ = \{\gamma \in \Gamma(m_2) : \gamma' = u^{m_2/2} \text{ and } \partial(u) \mid m\}$$

and

$$\tilde{\Gamma}(m)^- = \{\gamma \in \Gamma(m_2) : \gamma' = -u^{2^{\alpha-1}} \text{ and } \partial(u) \mid 2m \text{ but } \partial(u) \nmid m\}$$

Then

$$\tilde{\Gamma}(m) = \tilde{\Gamma}(m)^+ \cup \tilde{\Gamma}(m)^-$$

Proposition (Ali & FP - 2019 - preliminary version)

Let $\Gamma \subset \mathbb{Q}^+$ and $m \in \mathbb{N}$. Then $\delta_{\Gamma,m} = 0$ if one of the following holds:

1. $2 \nmid m \,\&\forall g \in \Gamma, \delta(g) \mid m$
2. $2 \mid m, 3 \nmid m, \Gamma(3) = \{1\} \,\&\exists g_0^{m_2/2}{\mathbb{Q}^*}^{m_2} \in \Gamma(m_2)$ s.t. $\delta(-3g_0) \mid m$
3. $2 \| m, 3 \nmid m, \Gamma(3) = \{1\}, \Gamma(2) \neq \{1\} \& \exists -g_0{\mathbb{Q}^*}^2 \in \Gamma(2), \delta(3g_0) \mid m$
4. $2 \| m, \Gamma(2) = \{1\}, \Gamma(4) \neq \{1\} \& \exists -g_0{\mathbb{Q}^*}^4 \in \Gamma(4), \delta(2g_0) \mid 2m$
5. $4 \| m, 3 \nmid m, \Gamma(6) = \{1\}, \Gamma(4) \neq \{1\} \& \exists -g_0{\mathbb{Q}^*}^4 \in \Gamma(4),$ $\delta(-6g_0) \mid 2m$
6. $3 \nmid m, \Gamma(3) = \{1\} \& \exists -g_0^{m_2/4}{\mathbb{Q}^*}^{m_2/2} \in \Gamma(m_2/2), \delta(-3g_0) \mid 2m$

Furthermore, we guess that one of these conditions is necessary in order to have $\delta_{\Gamma,m} = 0$.

If $\Gamma = \langle \alpha_1, \dots, \alpha_r \rangle \subset \mathbb{Q}^*$ and

$$\operatorname{supp}\Gamma = \{p : v_p(\alpha_i) \neq 0, \exists i = 1, \dots r\} = \{p_1, \dots, p_s\}$$

Let

$$E = E_{\alpha_1, \dots \alpha_t} := \left(e_{ij}\right)_{1 \le i \le r, 1 \le j \le s} \in M_{r,s}(\mathbb{Z})$$

defined by $a_j = p_1^{e_{j1}} \cdots p_s^{e_{js}}$

Definition

For $j \le r$, the *j–th elementary divisor*

$$\Delta_j = \Delta_j(\Gamma) = \operatorname{gcd} \text{all determinants of all the } j \times j \text{ minors of} E$$

Also $\Delta_j(\Gamma) := 0$ for $j > r$.

Given $\Gamma \subset \mathbb{Q}^*$ finitely generated of rank r, we have that for any $m \in \mathbb{N}$,

$$\#\Gamma(m) = \#\left(\Gamma \cdot (\mathbb{Q}^*)^m \big/ (\mathbb{Q}^*)^m\right) = \frac{m^r}{\gcd(m^r, m^{r-1}\Delta_1, \dots, m\Delta_{r-1}, \Delta_r)}$$

Andam Mustafa Ali

- To compute Δ_r use the Hermite normal form for E
- $\delta_{\Gamma,m} = \delta'_{\Gamma,m} \prod_\ell \left(1 - \frac{1}{\ell^r(\ell-1)}\right)$ with $\delta'_{\Gamma,m} \in \mathbb{Q}^{\geq 0}$
- Andam is writing GP-Pari scripts

  `https://pari.math.u-bordeaux.fr/Scripts/`
- They compute several invariants of $\Gamma = \langle a_1, \ldots, a_r \rangle$, $a_j \in \mathbb{Q}^*$. Among them also $\delta'_{\Gamma,m}$ and other related densities

Number Fields

- K number field, $\Gamma \subset K^*$ finitely generated
- $\operatorname{supp}\Gamma := \{\mathfrak{p}$ prime ideal of $\mathcal{O}_K : v_\mathfrak{p}(\alpha) \neq 0$ for some $\alpha \in \Gamma\}$.
- if $\mathfrak{p} \subset \mathcal{O}_K : \mathfrak{p} \notin \operatorname{supp}\Gamma$, $\Gamma_\mathfrak{p} = \{\gamma \bmod \mathfrak{p} : \gamma \in \Gamma\}$ is well defined
- $P_{\Gamma,m} = \{\mathfrak{p} \subset \mathcal{O}_K : \mathfrak{p} \notin \operatorname{supp}\Gamma, [\mathcal{O}_K/\mathfrak{p} : \Gamma_\mathfrak{p}] = m\}$
- Γ is *unit–free* if $\Gamma \cap \mathcal{O}_K^* = \operatorname{Tor}\Gamma$
- Γ unit–free $\Rightarrow \operatorname{rank}\Gamma \geq \#\operatorname{supp}\Gamma$.
- for Γ unit–free one can compute the elementary divisors
- Hence $\#\Gamma(m) = \#{K^*}^m\Gamma/{K^*}^m$