On interesting subsequences of the sequence of primes

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Collaborazione con János T. Tóth (János Sélye University in Komárno, Slovakia)

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$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

$$\mathbb{R}_+ = (0, +\infty)$$

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Let $A \subset \mathbb{N}$. Then

$$A(x) = \#(A \cap [1, x]), \quad x \geq 1,$$

$$R(A) = \{a/b : a, b \in A\},$$

$$R^d(A) = \text{ the set of accumulation points of } R(A) \subset \mathbb{R}_+.$$

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We say that A is (R)-dense if $R^d(A) = \mathbb{R}_+$.

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- The set of prime numbers in a fixed aritmetical progression with coprime coefficients is (R)-dense (Garcia, Selhorst-Jones, Poore, Simon, 2011).

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- The set of prime numbers in a fixed aritmetical progression with coprime coefficients is (R)-dense (Garcia, Selhorst-Jones, Poore, Simon, 2011).
- For each increasing and unbounded $f : \mathbb{R}_+ \to \mathbb{R}_+$ there exists an (R)-dense set $A \subset \mathbb{N}$ with $\lim_{x \to +\infty} A(x)/f(x) = 0$ (Hedman, Rose, 2009).

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$$\mathbb{P}_{k+1} \subsetneq \mathbb{P}_k$$

The sets $\mathbb{P}_k$

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Fact (Starni 1995; Hedman, Rose, 2009)

Let $A = \{a_1 < a_2 < \dots < a_n < \dots\} \subset \mathbb{N}$ such that $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 1$. Then the set A is (R)-dense.

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Corollary (M, Tóth)

The set $\mathbb{P}_k$ is (R)-dense for each $k \in \mathbb{N}_0$.

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Theorem (M, Tóth)

Let $k \in \mathbb{N}_0$. Then $\rho(\mathbb{P}_k) = 1$. Moreover, $\sum_{p \in \mathbb{P}_k} \frac{1}{p} < +\infty \Leftrightarrow k \geq 2$.

The sets $\mathbb{P}_n^T$

$$p_n^{(k+1)} \sim p_n^{(k)} \log p_n^{(k)} \text{ as } k \to +\infty.$$

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Fact (Tóth, Zsilinszki, 1995)

Let $A = \{a_1 < a_2 < \dots < a_n < \dots\} \subset \mathbb{N}$ with $\liminf_{n\to+\infty} \frac{a_{n+1}}{a_n} = c > 1$. Then $R^d(A) \cap (\frac{1}{c}, c) = \emptyset$.

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Corollary (M, Tóth)

For each $n \in \mathbb{N}$ the set $\mathbb{P}_n^T$ is not (R)-dense. Moreover, $R^d(\mathbb{P}_n^T) = \varnothing$.

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Theorem (M, Tóth)

Let $n, m \in \mathbb{N}$ with $n < m$. Then,

$$\mathbb{P}_n^T \cap \mathbb{P}_m^T \neq \emptyset \iff m \in \mathbb{P}_n^T .$$

Moreover, if $m \in \mathbb{P}_n^T$, then $\mathbb{P}_m^T \subset \mathbb{P}_n^T$ and

$$\mathbb{P}_n^T \setminus \mathbb{P}_m^T = \{p_n^{(1)}, p_n^{(2)}, \ldots, p_n^{(k)}\} \quad \text{where} \quad m = p_n^{(k)} .$$

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For any $m, n \in \mathbb{N}$, $n < m$ we have

$$\left|\mathbb{P}_m^T(x) - \mathbb{P}_n^T(x)\right| \in \{j, j+1\}, ,$$

where $p_n^{(j)} \leq m < p_n^{(j+1)}$. Then,

$$\mathbb{P}_m^T(x) \sim \mathbb{P}_n^T(x) .$$

Bounds for $p_n^{(j)}$ (n fixed)

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(i) $p_n^{(j)} \geq p_n^{(k)} \log^{j-k} p_n^{(k)}$,

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Corollary (M, Tóth)

Let $n \in \mathbb{N}$ and $c \in (0,1)$. Then, we have

$$\mathbb{P}_n^T(x) = o(\log x) \text{ and } \mathbb{P}_n^T(x) = \omega(\log^c x)$$

as $x \to +\infty$. In particular,

$$\log \mathbb{P}_n^T(x) \sim \log\log x.$$

Bounds for $\mathbb{P}_n^T$ and convergence exponent for $\mathbb{P}_n^T$

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Theorem (M, Tóth)

For every $n \in \mathbb{N}$ we have $\rho(\mathbb{P}_n^T) = 0$. For each $\alpha > 0$ $S_n^{T,\alpha} = \sum_{p \in \mathbb{P}_n^T} \frac{1}{p^\alpha} \to 0$ as $n \to +\infty$. Moreover, if we put $S_n^{T,\alpha}(x) = \sum_{p \in \mathbb{P}_n^T, p \le x} \frac{1}{p^\alpha}$ and assume that $p_n^{(k-1)} \le x < p_n^{(k)}$ for some integer $k \ge 2$, then $S_n^{T,\alpha} - S_n^{T,\alpha}(x) \le \frac{1}{\left(p_n^{(k)}\right)^\alpha} \frac{\left(\log p_n^{(k)}\right)^\alpha}{\left(\log p_n^{(k)}\right)^\alpha - 1}$.

Let us consider the set Diag$\mathbb{P} = \{p_k^{(k)} : k \in \mathbb{N}\}$ of the elements on the diagonal of the infinite matrix

$$[p_n^{(k)}]_{n,k\in\mathbb{N}} = \begin{bmatrix} p_1^{(1)} & p_1^{(2)} & \dots & p_1^{(k)} & \dots \\ p_2^{(1)} & p_2^{(2)} & \dots & p_2^{(k)} & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ p_n^{(1)} & p_n^{(2)} & \dots & p_n^{(k)} & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}.$$

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Properties of Diag$\mathbb{P}$ are analogous to those of $\mathbb{P}_n^T$, $n \in \mathbb{N}$.

$$p_{k+1}^{(k+1)} > p_{k+1}^{(k)} \log p_{k+1}^{(k)} > p_k^{(k)} \log p_k^{(k)}.$$

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For each $n \in \mathbb{N}$ the set Diag$\mathbb{P}$ is not (R)-dense. Moreover, $R^d(\text{Diag}\mathbb{P}) = \varnothing$.

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Theorem (M, Tóth)

We have $\rho(\text{Diag}\mathbb{P}) = 0$. Moreover, if we put $S^\alpha_{diag} = \sum_{p \in \text{Diag}\mathbb{P}} \frac{1}{p^\alpha}$ and $S^\alpha_{diag}(x) = \sum_{p \in \text{Diag}\mathbb{P}, p \leq x} \frac{1}{p^\alpha}$ and assume that $p_{k-1}^{(k-1)} \leq x < p_k^{(k)}$ for some integer $k \geq 2$, then

$$S^\alpha_{diag} - S^\alpha_{diag}(x) \leq \frac{1}{\left(p_k^{(k)}\right)^\alpha} \frac{\left(\log p_k^{(k)}\right)^\alpha}{\left(\log p_k^{(k)}\right)^\alpha - 1}.$$

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Theorem (M, Tóth)

For each $k, j \in \mathbb{N}$ with $j \geq k$ the following estimations hold:

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Theorem (M, Tóth)

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Corollary (M, Tóth)

There exists a $k_0 \in \mathbb{N}$ such that for each $k, j \in \mathbb{N}$, $j \geq k \geq k_0$, we have

$$p_j^{(j)} \leq 8^{j-k} p_k^{(k)} \prod_{i=1}^{j-k} \left(\log p_k^{(k)} + i \log i \log \log p_k^{(k)} \right)^2 .$$

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Corollary (M, Tóth)

We have

$$1 \leq \liminf_{j \to +\infty} \frac{\log p_j^{(j)}}{j \log j} \leq \limsup_{j \to +\infty} \frac{\log p_j^{(j)}}{j \log j} \leq 2.$$

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Corollary (M, Tóth)

Let $n \in \mathbb{N}$ and $c \in (0,1)$. Then, we have

$$\mathrm{Diag}\mathbb{P}(x) = o(\log x) \text{ and } \mathrm{Diag}\mathbb{P}(x) = \omega(\log^c x)$$

as $x \to +\infty$. In particular,

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Corollary (M, Tóth)

The following inequalities hold for each $n \in \mathbb{N}$.

$$1 \leq \liminf_{x \to +\infty} \frac{\mathbb{P}_n^T(x)}{\mathrm{Diag}\mathbb{P}(x)} \leq \limsup_{x \to +\infty} \frac{\mathbb{P}_n^T(x)}{\mathrm{Diag}\mathbb{P}(x)} \leq 2.$$

In particular, $\mathrm{Diag}\mathbb{P}(x) = \theta\left(\mathbb{P}_n^T(x)\right)$.

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Answer: NO.

Solution of problems

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Theorem (Żmija)

For each $n \in \mathbb{N}$ we have

$$\mathbb{P}_n^T(x) \sim \mathrm{Diag}\mathbb{P}(x) \sim \frac{\log x}{\log\log x}.$$

Further problems

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Proof (Sanna).

Let $k > p_n$. Then

$$0 < \frac{p_n^{(k)}}{p_k^{(k)}} < \frac{p_n^{(k)}}{p_{p_n}^{(k)}} < \frac{p_n^{(k)}}{p_n^{(k+1)}} < \frac{1}{\log p_n^{(k)}}.$$

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where $\phi : [1, +\infty) \to [1, +\infty)$ is non-decreasing and such that:

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The properties of the sets A_k, A_n^T and Diag$A$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, are analogous as for the sets $\mathbb{P}_k$, $\mathbb{P}_n^T$ and Diag$\mathbb{P}$, respectively. (Joint work in progress with Jan Šustek (Ostrava, Czech Republic) & János T. Tóth).

Grazie mille!