Drinfeld cusp forms of prime level: structure and slopes

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joint with A. Bandini

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Classical modular forms

$N, k \in \mathbb{N}_{\geqslant 0}$, $p \in \mathbb{Z}$ prime

$S_k(\Gamma_0(N))$ cusp forms of level N and weight k

Hecke operators $\mathbf{T}_p$, $p \in \mathbb{Z}$ if $p \nmid N$

Atkin-Lehner operator $\mathbf{U}_p$ if $p|N$.

Fourier expansion $f = \sum_{n \geqslant 1} a_n(f) q^n$, $a_n(f) \in \mathbb{C}$.

Fix $p \nmid N$, $f \in S_k(\Gamma_0(pN))$ eigenform, then $\mathbf{U}_p f = a_p(f) f$.

The *p-slope* of f is $v_p(a_p(f))$.

We have $S_k(\Gamma_0(N)) \hookrightarrow S_k(\Gamma_0(pN))$ via maps

$$\delta_1, \delta_p : S_k(\Gamma_0(N)) \hookrightarrow S_k(\Gamma_0(pN))$$

$$\delta_1(f)(z) = f(z) \quad \text{and} \quad \delta_p(f)(z) = f(pz).$$

Oldforms: all cusp forms generated by $Im(\delta_1)$ and $Im(\delta_p)$.

Newforms: orthogonal complement of oldforms w.r.t. the Petersson inner product.

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\mathbb{Z} & \leftrightarrow & \mathcal{O} = \mathbb{F}_q[t] \\
\mathbb{C} & \leftrightarrow & \mathbb{C}_\infty = \widehat{\overline{K}}_\infty, \quad \infty = \frac{1}{t} \\
SL_2(\mathbb{Z}) & \leftrightarrow & GL_2(\mathcal{O}) \\
\mathcal{H} & \leftrightarrow & \Omega := \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(K_\infty) \\
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For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(K_\infty)$, $k, m \in \mathbb{Z}$ and $\varphi : \Omega \to \mathbb{C}_\infty$, we define

$$(\varphi|_{k,m}\gamma)(z) := \varphi(\gamma z)(\det \gamma)^m (cz+d)^{-k}.$$

Let Γ be any congruence subgroup of $GL_2(\mathcal{O})$.

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Definition

A rigid analytic function $\varphi : \Omega \to \mathbb{C}_\infty$ is called a Drinfeld modular form of weight k and type m for Γ if

- *φ is holomorphic on Ω and at all cusps;*
- $(\varphi|_{k,m}\gamma)(z) = \varphi(z) \quad \forall \gamma \in \Gamma$

A Drinfeld modular form φ is called a cusp form if it vanishes at all cusps.

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Atkin-Lehner operator $\mathbf{U}_\mathfrak{p}$ if the level is $\mathfrak{p}\mathfrak{m}$

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Fourier expansion (technical): depends on the Carlitz exponential e_C with period $\tilde{\pi}$

$$\varphi(z) = \sum_{n \geqslant 0} a_n(\varphi) \frac{1}{e_C(\tilde{\pi} z)^n}, \; a_n(\varphi) \in \mathbb{C}_\infty$$

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- Old eigenforms come in pairs and the p-slopes of a pair add up to $k-1$.
  Indeed, let $f \in S_k(\Gamma_0(N))$ be an eigenform, then $\mathbf{U}_p$ acts on $< \delta_1(f), \delta_p(f) >$ with characteristic polynomial

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- Analogue of Petersson inner product is not available.

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Definition

The space of oldforms of level $\mathfrak{m}$, *denoted by* $S^{1,old}_{k,m}(\Gamma_0(\mathfrak{m}))$, *is the subspace of* $S^1_{k,m}(\Gamma_0(\mathfrak{m}))$ *generated by the set*

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is injective. Moreover, we have an equality of sets

$$\{\text{Eigenvalues of } \mathbf{U}_{\mathfrak{p}}{}_{|S^{1,old}_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))}\} = \{\text{Eigenvalues of } \mathbf{T}_{\mathfrak{p}}\} \cup \{0\}.$$

Now we focus on the case $\mathfrak{m} = (1)$.
System of representatives for $\Gamma_0(\mathfrak{p})\backslash GL_2(\mathcal{O})$ given by

$$R = \left\{ Id, \begin{pmatrix} 0 & -1 \\ 1 & Q \end{pmatrix} \text{ s.t. } Q \in \mathcal{O} \text{ and } \deg Q < d \right\}.$$

Definition

We have the following maps defined on $S^1_{k,m}(\Gamma_0(\mathfrak{p}))$:

- *the Fricke involution, which preserves the space $S^1_{k,m}(\Gamma_0(\mathfrak{p}))$, is represented by the matrix*

$$\gamma_{\mathfrak{p}} := \begin{pmatrix} 0 & -1 \\ P_d & 0 \end{pmatrix}$$

  and defined by $\varphi^{Fr} = (\varphi|_{k,m}\gamma_{\mathfrak{p}})$;
- *the trace map is defined by*

$$Tr : S^1_{k,m}(\Gamma_0(\mathfrak{p})) \to S^1_{k,m}(GL_2(\mathcal{O}))$$
$$\varphi \mapsto \sum_{\gamma \in R} (\varphi|_{k,m}\gamma)(z);$$

- *the twisted trace map is defined by*

$$Tr' : S^1_{k,m}(\Gamma_0(\mathfrak{p})) \to S^1_{k,m}(GL_2(\mathcal{O}))$$
$$\varphi \mapsto Tr(\varphi^{Fr}).$$

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$$\psi_1 := \delta_1\varphi - \frac{P_d^{k-m}}{\lambda}\delta_\mathfrak{p}\varphi \ \in Ker(Tr)$$

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Conjectures (Special case $P_1 = t$)

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There exists a basis for $S_k^1(\Gamma_1(t))$ (using combinatorial techniques) provided by

$$\mathcal{B} = \{\mathbf{c}_j \mid 0 \leqslant j \leqslant k-2.\}$$

The action of $\mathbf{U}_t$ on this basis is given by

$$\mathbf{U}_t(\mathbf{c}_j) = -(-t)^{j+1}\binom{k-2-j}{j}\mathbf{c}_j - t^{j+1}\sum_{h\neq 0}\left[\binom{k-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1}\binom{k-2-j-h(q-1)}{j}\right]\mathbf{c}_{j+h(q-1)}$$

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Let $C_j = \langle \mathbf{c}_s : s \equiv j \pmod{q-1}\rangle$ with $0 \leqslant j \leqslant q-2$, then the C_j are stabe for the action of $\mathbf{U}_t$. Reordering the basis we have a block matrix which is diagonalizable $\Longleftrightarrow$ each block is.

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Blocks arising from $S^1_{k,m}(\Gamma_0(t))$: there is a natural inclusion $S^1_{k,m}(\Gamma_0(t)) \hookrightarrow S^1_k(\Gamma_1(t))$ and the subspaces arising from $\Gamma_0(t)$ are those for $k \equiv 2(j+1) \pmod{q-1}$.

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The associated coefficient matrix (i.e. with entries in $\mathbb{F}_p$)

$$M = \begin{pmatrix}
m_{1,1} & m_{1,2} & \cdots & m_{1,\frac{n}{2}} & (-1)^{j+1}m_{1,\frac{n}{2}} & \cdots & (-1)^{j+1}m_{1,2} & (-1)^{j+1}(m_{1,1}-1) \\
m_{2,1} & m_{2,2} & \cdots & m_{2,\frac{n}{2}} & (-1)^{j+1}m_{2,\frac{n}{2}} & \cdots & (-1)^{j+1}(m_{2,2}-1) & (-1)^{j+1}m_{2,1} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
m_{\frac{n}{2},1} & m_{\frac{n}{2},2} & \cdots & m_{\frac{n}{2},\frac{n}{2}} & (-1)^{j+1}(m_{\frac{n}{2},\frac{n}{2}}-1) & \cdots & (-1)^{j+1}m_{\frac{n}{2},2} & (-1)^{j+1}m_{\frac{n}{2},1} \\
m_{\frac{n}{2}+1,1} & m_{\frac{n}{2}+1,2} & \cdots & (-1)^j & 0 & \cdots & (-1)^{j+1}m_{\frac{n}{2}+1,2} & (-1)^{j+1}m_{\frac{n}{2}+1,1} \\
\vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
m_{n-1,1} & (-1)^j & \cdots & 0 & 0 & \cdots & 0 & (-1)^{j+1}m_{n-1,1} \\
(-1)^j & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$

The dimension n comes from the equality $k = 2(j+1) + (n-1)(q-1)$.

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where for $1 \leqslant i \leqslant n$ we set $s_i = j + 1 + (i-1)(q-1)$.

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If $\dim_{\mathbb{C}_\infty} S^1_{k,m}(GL_2(\mathcal{O})) \leqslant 1$, *then all conjectures hold.*

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Further conjectures

We also formulated conjectures on the distribution of t-slopes that can be related to the existence of families of Drinfeld modular forms (see also the work of S. Hattori).

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Thanks!