Minimizing GCD sums and applications
Number Theory Meeting, Torino

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Introduction

In 1949 Gál introduced the following sums associated to a set $\mathcal{M}$ and defined by

$$S_\alpha(\mathcal{M}) := \sum_{m_1, m_2 \in \mathcal{M}} \frac{(m_1, m_2)^{2\alpha}}{(m_1 m_2)^\alpha} \qquad (0 < \alpha \leqslant 1)$$

where (m_1, m_2) denotes the greatest common divisor of m_1 and m_2.

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- Recently, new interest in connection with large values of the Riemann zeta function.

Maximum of GCD sums

<u>Question</u>: What is the maximal size (in terms of N) of $S_\alpha(\mathcal{M})$ among all the choices of sets $\mathcal{M}$ of a fixed size N?

- For $1/2 < \alpha \leq 1$, optimal results (Gál, Aistleitner-Berkes-Seip).
- $\max_{|\mathcal{M}|=N} \frac{S_{1/2}(\mathcal{M})}{|\mathcal{M}|} \geq \exp\left\{(C+o(1))\sqrt{\frac{\log N \log_3 N}{\log_2 N}}\right\}$ where $\log_k$ is the kth-iterative of the logarithm(Bondarenko-Seip)
- Optimal constant $C = 2\sqrt{2}$ obtained by de la Bretèche and Tenenbaum.

<u>Consequence:</u>

$\max_{t\in[0,T]} |\zeta(\frac{1}{2}+it)| \geq \exp\left\{(2\sqrt{2}+o(1))\sqrt{\frac{\log T \log_3 T}{\log_2 T}}\right\}$

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What about small GCD sums?

In the maximum problem, the cardinality of $\mathcal{M}$ is fixed while the size of its elements is not (allow to choose very sparse sets).

We study the minimal value of the ratio

$$\mathcal{T}(N) := \inf_{w \in (\mathbb{R}_+)^N} \left(\frac{N}{||w||_1^2} \sum_{m_1, m_2 \leqslant N} w(m_1) w(m_2) \frac{(m_1, m_2)}{\sqrt{m_1 m_2}} \right). \quad (1)$$

Example: When $w(m) \in \{0, 1\}$, equivalent to

$$\inf_{\mathcal{M} \subset [1, N]} \left(\frac{N}{|\mathcal{M}|^2} \sum_{m_1, m_2 \in \mathcal{M}} \frac{(m_1, m_2)}{\sqrt{m_1 m_2}} \right). \quad (2)$$

Maximal density of a set $\mathcal{M}$ such that

$$\sum_{m_1, m_2 \in \mathcal{M}} \frac{(m_1, m_2)}{\sqrt{m_1 m_2}} \ll |\mathcal{M}| (\log |\mathcal{M}|)^{o(1)}$$

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Main result and applications

Trivial bound: $\mathcal{T}(N) \ll \log N$ (take $\mathcal{M} = \{1, \ldots, N\}$ or $\mathcal{M} = \{p \leq N, \text{ p prime}\}$).

Theorem (La Bretèche, M., Tenenbaum 2019)

There exists $\eta \approx 0.16656 < 1/6$ *such that when* N *tends to* $+\infty$

$$(\log N)^\eta \ll \mathcal{T}(N) \ll (\log N)^\eta (\log_2 N)^3.$$

Application:

- Logarithmic improvements of the Burgess' bound on multiplicative character sums
- Non-vanishing of theta functions
- Lower bounds on small moments of character sums

A related minimization problem gives better results in last two applications.

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Multiplicative character sums

Let us consider

$$S_\chi(M,N) := \sum_{M<n\leqslant M+N} \chi(n),$$

where $\chi \mod p$ is a multiplicative character.

Example: The Legendre symbol $n \to \left(\frac{n}{p}\right)$.

Question: How large N should be to ensure $S_\chi(M,N) = o(N)$?

Applications: Distribution of quadratic residues modulo p, primitive roots of $\mathbb{F}_p^*$ etc

$$|S_\chi(M,N)| \ll \sqrt{p}\log p \qquad \text{Pólya and Vinogradov.} \tag{3}$$

- Non trivial when $N > p^{1/2+\varepsilon}$.

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Burgess' bound

Major breakthrough obtained by Burgess in 1962

$$|S_\chi(M,N)| \ll N^{1-1/r} p^{(r+1)/4r^2} \log p, \qquad r \geq 2. \tag{4}$$

- Non trivial when $N \geq p^{1/4+\varepsilon}$.
- Iwaniec-Kowalski $(\log p)^{1/2r}$
- Kerr-Shparlinski-Yau $(\log p)^{1/4r}$

As an application of GCD sums, we prove:

Theorem (La Bretèche, M., Tenenbaum 2019)

$$S_\chi(M,N) \ll N^{1-1/r} p^{(r+1)/4r^2} (\log p)^{(\eta+o(1))/2r}, \qquad \eta \approx 0.16656.$$

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Multiplicative energy and a related problem

For two sets $\mathcal{A}, \mathcal{B} \subset [1, N]$, we consider the multiplicative energy

$$E_\times(\mathcal{A}, \mathcal{B}) := \left| \{ m_1, n_1 \in \mathcal{A}, m_2, n_2 \in \mathcal{B} \, : \, m_1 m_2 = n_1 n_2 \} \right|.$$

- Appears to be of great importance in additive combinatorics.
- GCD sums are related to the quantity

$$E_\times(N, \mathcal{B}) := \left| \{ 1 \leq m_1, n_1 \leq N, m_2, n_2 \in \mathcal{B} \, : \, m_1 m_2 = n_1 n_2 \} \right|.$$

In view of our applications, we need to bound $E_\times(\mathcal{B}, \mathcal{B})$.
Analogous question: How dense can we choose $\mathcal{B} \subset [1, N]$ such that $E_\times(\mathcal{B}, \mathcal{B}) \ll |\mathcal{B}|^2 (\log N)^{o(1)}$.

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Analogous question: How dense can we choose $\mathcal{B} \subset [1, N]$ such that $E_\times(\mathcal{B}, \mathcal{B}) \ll |\mathcal{B}|^2 (\log N)^{o(1)}$.

Minimization problem and main result

This is equivalent to estimate the quantity

$$\mathcal{E}_N := \inf_{\mathcal{B}\subset[1,N]} N^2 E_\times(\mathcal{B},\mathcal{B})/|\mathcal{B}|^4.$$

Theorem (La Bretèche, M., Tenenbaum 2019)

Let $\delta := 1 - \frac{1+\log_2 2}{\log 2} \approx 0.08607$*. When* N *tends to* $+\infty$*, we have*

$$(\log N)^\delta (\log_2 N)^{3/2} \ll \mathcal{E}_N \ll (\log N)^\delta \big(\log_2 N\big)^6.$$

The exponent δ is the one appearing in the famous multiplication table problem of Erdős:

$$H(N) = \big|\{n \le N^2 \ \exists a, b \le N,\ n = ab\}\big| \asymp \frac{N^2}{(\log N)^{\delta+o(1)}}.$$

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Small moments of random multiplicative functions and character sums

$p \to X(p)$ a random variable uniformly distributed on $\{z, |z| = 1\}$.
Random multiplicative function: $X(n) = \prod_{p^\alpha \| n} X(p)^\alpha$.

Conjecture(Helson):

$$\mathbb{E}|\sum_{n \le N} X(n)| = o(\sqrt{N}).$$

- Harper (2018) proved the conjecture under the following form $\mathbb{E}\left|\sum_{n \le N} X(n)\right| \asymp \frac{\sqrt{N}}{(\log\log N)^{1/4}}$.
- Deterministic analogues?

We have more than square-root cancellation (Harper, announced)

$$\frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{n \leqslant N} \chi(n) \right| \ll \frac{\sqrt{N}}{\min\left\{(\log\log L)^{1/4}, (\log\log\log p)^{1/4}\right\}}$$

where $L = \min\{N, p/N\}$.

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Lower bounds for moments of character sums

Obtaining good lower bounds from the probabilistic methods used by Harper seems hard.

Theorem (La Bretèche, M., Tenenbaum 2019)

For p sufficiently large and $L := \min(N, p/N)$, we have

$$\frac{1}{p-2} \sum_{\chi \neq \chi_0} \left| \sum_{n \leqslant N} \chi(n) \right| \gg \sqrt{\frac{N}{\mathcal{E}_L}} \gg \frac{\sqrt{N}}{(\log L)^{\delta/2} (\log_2 L)^3}$$

with $\delta/2 \approx 0.043$.

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Choice of $\mathcal{M}$ and rough idea of the proof

Let $N_k(N)$ denote the number of integers $n \leqslant N$ such that $\Omega(n) = k$ (where $\Omega(n)$ denote the total number of prime factors of n, counted with multiplicity).

We have for $\kappa := k/\log_2 N$

$$N_k(N) \asymp \frac{N}{(\log N)^{Q(\kappa)}\sqrt{\log_2 N}}$$

where $Q(x) = x\log x - x + 1$.

To prove the upper bound for $\mathcal{T}(N)$, we choose the set of integers $n \in (N/2, N)$ such that $\Omega(n) = \kappa \log_2 N$ and

$$\Omega(n,t) := \sum_{p^\nu \| n,\, p\leqslant t} \nu \leqslant \kappa \log_2 3t + C (1 \leqslant t \leqslant N).$$

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Local properties of divisors

Denote by $F_k(N; C)$ the number of integers $n \leqslant N$ counted by $N_k(N)$ and such that

$$\Omega(n,t) \leqslant \kappa \log_2 3t + C \quad (1 \leqslant t \leqslant N) \tag{5}$$

with $\kappa := k/\log_2 x$.

Lemma (Deduced from the work of Ford)

Let $\kappa_0 \in]0,2[$. For $0 \leqslant \kappa \leqslant \kappa_0$ and suitable $C = C(\kappa_0)$, we have

$$F_k(N; C) \asymp \frac{N_k(N)}{k} \quad (N \geqslant 3). \tag{6}$$

Thank you for your attention!