A lower bound for the variance of generalized divisor functions in arithmetic progressions

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The variance of arithmetic functions in arithmetic progressions

Let f be a complex arithmetic function. We “generally expect” uniformly distribution in arithmetic progressions

$$\sum_{\substack{n\leq N \\ n\equiv a \pmod q}} f(n) \approx \frac{1}{\phi(q)} \sum_{\substack{n\leq N \\ (n,q)=1}} f(n),$$

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for any $a \pmod q$ with $(a,q)=1$. In order to understand whether this point of view may be correct or not we study the variance

$$V_q(f) = \frac{1}{\phi(q)} \sum_{\substack{a=1,\ldots,q \\ (a,q)=1}} \Bigg| \sum_{\substack{n\leq N \\ n\equiv a \pmod q}} f(n) - \frac{1}{\phi(q)} \sum_{\substack{n\leq N \\ (n,q)=1}} f(n) \Bigg|^2.$$

Some results about $V_q(f)$

- Elliott and Hildebrand: upper bounds for $V_q(f)$, when f is a multiplicative function with $|f| \leq 1$.
- Balog, Granville and Soundararajan: asymptotic for $V_q(f)$ as before.
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- Klurman, Mangerel and Teräväinen: upper bound for variance of 1–bounded multiplicative functions in short arithmetic progressions.
- Banks, Heath–Brown and Shparlinski: asymptotic for variance of $d_2(n) = \sum_{d|n} 1$.
- Keating, Rodgers, Roditty–Gershon and Rudnick: conjecture on asymptotic for variance of $d_k(n) = \sum_{e_1 e_2 \cdots e_k = n} 1$.

An averaged version of the variance

Definition

We define the variance of f in arithmetic progressions by

$$V(Q,f) = \sum_{q \leq Q} V(q,f), \tag{1}$$

where

$$V(q,f) = \sum_{h|q} \sum_{\substack{a \mod q \\ (a,q)=h}} \Bigg| \sum_{\substack{n \leq N \\ n \equiv a \mod q}} f(n) - \frac{1}{\phi(q/h)} \sum_{\substack{n \leq N \\ (n,q)=h}} f(n) \Bigg|^2. \tag{2}$$

Previous results on $V(Q,f)$

- Motohashi: asymptotic equality for $V(Q, d_2)$.
- Rodgers and Soundararajan: asymptotic equality for $V(Q, d_k, \phi)$, where $d_k(n)$ is twisted with a smooth weight $\phi(t)$.
  Important to note: it deals only with values of Q lying in a limited range, specifically between two powers of N depending on k.
- Harper and Soundararajan: lower bound for $V(Q, d_k)$, for any Q in the range $N^{1/2+\delta} \leq Q \leq N$, with $\delta > 0$ small, and any $k \geq 2$ a positive integer.

A lower bound for $V(Q, d_\alpha)$

We define the α–fold divisor function $d_\alpha(n)$ as the n-th coefficient in the Dirichlet series of $\zeta(s)^\alpha$ on the half plane $\Re(s) > 1$, so that in such region we have

$$\zeta(s)^\alpha = \sum_{n \geq 1} \frac{d_\alpha(n)}{n^s}.$$

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Theorem (M.)

Let $\delta > 0$ sufficiently small and consider $N^{1/2+\delta} \leq Q \leq N$. For any complex number $\alpha \notin -\mathbb{N} \cup \{0, 1\}$, we have

$$V(Q, d_\alpha) \gg_{\alpha,\delta} Q \sum_{n\leq N} |d_\alpha(n)|^2, \tag{3}$$

if N large enough with respect to α and δ.

Connecting the variance with the L^2-norm of exponential sums

Given parameters Q_0, Q and K, we are going to define the so called set of major arcs

$$\mathfrak{M} = \{\varphi \in \mathbb{R}/\mathbb{Z} : |\varphi - a/q| \leq K/(qQ), \text{ with } q \leq KQ_0 \text{ and } (a,q) = 1\}.$$

Let $\mathfrak{m}$, the minor arcs, denote the complement of the major arcs in $\mathbb{R}/\mathbb{Z}$.

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More similar to the Hooley's approach than previous works of Liu and Perelli, where instead the connection was between character sums and exponential sums.

Harper and Soundararajan proved the following main proposition which makes explicit such connection.

Proposition

Let N be large, let K be a large constant, and let Q_0 and $K\sqrt{N\log N} \leq Q \leq N$ be such that

$$\frac{N\log N}{Q} \leq Q_0 \leq \frac{Q}{K^2}. \tag{4}$$

We then have

$$V(Q, d_\alpha) \gg Q\int_{\mathfrak{m}} |\mathcal{F}(\varphi)|^2 d\varphi + Error \tag{5}$$

where $\mathcal{F}(\varphi) := \sum_{n\leq N} d_\alpha(n)e(n\varphi)$.

Heuristic for the variance growth

If the contribution of $\mathcal{F}(\varphi)$ on the minor arcs exceeds that on the major arcs, we can approximate the integral in (5) with the integral over all the circle, obtaining

$$\int_{\mathfrak{m}} |\mathcal{F}(\varphi)|^2 d\varphi \approx \sum_{n \leq N} |d_\alpha(n)|^2.$$

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- We believe it for any $d_\alpha(n)$ with $\alpha \neq 1$.
- Negative values, e.g. $\alpha = -1$, situation not clear: $\mathcal{F}(\varphi)$ over the major arcs would be close to $\sum_{n\leq N} \mu(n)$, which by the squareroot cancellation principle it is at most $N^{1/2+\varepsilon}$. On the other hand, over the minor arcs, $\mathcal{F}(\varphi)$ is affected by the random fluctuations of the prime numbers, suggesting a value of roughly $\sqrt{N}$, being a sum of N pseudorandom phases.

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- $\alpha = 1$: it does not hold.

The case of $\alpha = 0$ and $\alpha = 1$

The simple form of $d_0(n)$ and $d_1(n)$ allows us to elementarily study the associated variance.

Proposition

For any $Q \geq 1$, we have

$$V(Q, d_0) = Q + O(\log Q),$$
$$V(Q, d_1) \ll Q^2.$$

In particular, the above result shows that the value of $V(Q, d_1)$ does not match the Parseval heuristic.

Lower bounding the integral over the minor arcs

We apply the Cauchy–Schwarz inequality to get

$$\int_{\mathfrak{m}} |\mathcal{F}(\varphi)|^2 d\varphi \geq \left(\int_{\mathfrak{m}} |\mathcal{F}(\varphi)\tilde{\mathcal{F}}(\varphi)| d\varphi \right)^2 \left(\int_{\mathfrak{m}} |\tilde{\mathcal{F}}(\varphi)|^2 d\varphi \right)^{-1}, \quad (6)$$

where $\tilde{\mathcal{F}}(\varphi)$ is the exponential sums with coefficients $\sum_{r\leq R, r|n} d_{\alpha-1}(r) \approx d_\alpha(n)$, for a suitable $R \approx \sqrt{N}$.

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Proposition (Harper–Soundararajan)

We have

$$\int_{\mathfrak{m}} |\mathcal{F}(\varphi)\tilde{\mathcal{F}}(\varphi)| d\varphi \gg \sum_{KQ_0 < q \leq R} \Big| \sum_{\substack{r\leq R \\ q|r}} \frac{d_{\alpha-1}(r)}{r} \Big| \Big| \sum_{n\leq N} d_\alpha(n) c_q(n) \Big| + Error, \qquad (7)$$

where

$$c_q(n) = \sum_{1\leq a\leq q,\ (a,q)=1} e(an/q).$$

Main difference with Harper and Soundararajan work

We need to find a lower bound for $|\sum_{n\leq N} d_\alpha(n)c_q(n)|$.

- $\alpha \in \mathbb{N}_{\geq 2} \longrightarrow$ Perron's formula (approach used by Harper and Soundararajan). Admissible strategy because the Dirichlet series corresponding to $d_\alpha(n)c_q(n)$ is a slight variation of $\zeta(s)^\alpha$; since α is a positive integer, it can be extended as a meromorphic function on the whole complex plane with just one pole at 1.

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- $\alpha \in \mathbb{C} \longrightarrow$ elementary analysis of the sum using properties of the Ramanujan sums, since the function $\zeta(s)^\alpha$ may have an essential singularity at $s = 1$.

Strategy of the proof of the Theorem

- We may rewrite the main sum as

$$\sum_{n\leq N} d_\alpha(n)c_q(n) = \sum_{\substack{b\leq N \\ p|b\Rightarrow p|q}} d_\alpha(b)c_q(b) \sum_{\substack{a\leq N/b \\ (a,q)=1}} d_\alpha(a), \tag{8}$$

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  using the unique substitution $n = ab$, with $(a,q) = 1$ and $b = n/a$.
- Restricting the set of integers q we are working with and using the Selberg–Delange's method, we get

$$\sum_{\substack{a\le N/b\\ (a,q)=1}} d_\alpha(a) = c(\alpha)\frac{N(\log(N/b))^{\alpha-1}}{b} + Error, \tag{9}$$

  for a certain constant $c(\alpha) \in \mathbb{C} \setminus \{0\}$.

Putting things together we have the following main term

$$Nc(\alpha) \sum_{b \le N:\ p|b \Rightarrow p|q} \frac{d_\alpha(b) c_q(b)}{b} (\log(N/b))^{\alpha-1},$$

which we can show to be roughly

$$N(\log N)^{\alpha-1} c(\alpha) \sum_{b|q} d_\alpha(b) \mu(q/b) = N(\log N)^{\alpha-1} c(\alpha) d_{\alpha-1}(q). \qquad (10)$$

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- Dirichlet's hyperbola method to separate different contributions coming from the sum over b.
- further decomposition of the Ramanujan sum.
- assuming $q = rst$, where:
  - s is $(\log N)^B$–smooth, for a large constant $B > 0$.
  - $r \le N^\varepsilon$, for a small $\varepsilon > 0$, and $(\log N)^B$–rough.
  - t is a large prime close in size to $\sqrt{N}$.
  - other technical hypotheses on the arithmetical structure of q.

Conclusion of the proof

We end up with

$$|\sum_{n\leq N} d_\alpha(n)c_q(n)| \gg N(\log N)^{\Re(\alpha)-1}|d_{\alpha-1}(q)|,$$

from which we deduce

$$V(Q,d_\alpha) \gg QN(\log N)^{-|\alpha-1|^2+2(\Re(\alpha)-1)}\left(\sideset{}{^*}\sum_{q\leq N}\frac{|d_{\alpha-1}(q)|^2}{q}\right)^2 \gg QN(\log N)^{|\alpha|^2-1} \sim Q\sum_{n\leq N}|d_\alpha(n)|^2, \quad (11)$$

as $N\to\infty$, again by Selberg–Delange's theorem, which in turn concludes the proof of the Theorem.

Generalizations and corollaries

The new elementary method introduced is quite flexible and allows for several interesting generalizations. In particular, we are able to extend the result to a wide class of multiplicative functions that behave like a divisor function on average over prime numbers in a suitable way.
As a corollary of this, for instance we can show a lower bound for the variance in arithmetic progressions of the indicator function of the sums of two squares.
Moreover, even though we do not expect to generate a lower bound for the variance of d_1 over arithmetic progressions using the above approach, we are still able to produce one for a suitable sequence of divisor functions, depending on N, converging to d_1, as $N \to \infty$.

THANK YOU!