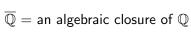
Images of Galois representations

Andrea Conti

University of Luxembourg

October 25, 2019



 $\overline{\mathbb{Q}}$ = an algebraic closure of \mathbb{Q}

We study the profinite group $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via its continuous representations:

 $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_n(A)$

where A is a topological ring.

Consider an elliptic curve E defined over \mathbb{Q} .

(

Consider an elliptic curve E defined over \mathbb{Q} .

 $E(\overline{\mathbb{Q}})$ is equipped with a group structure.

Consider an elliptic curve E defined over \mathbb{Q} .

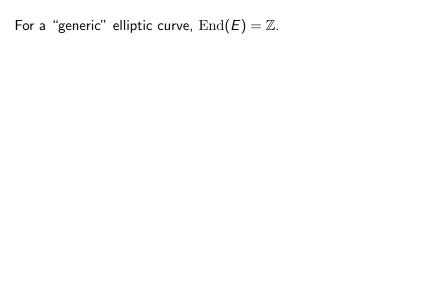
 $E(\overline{\mathbb{Q}})$ is equipped with a group structure.

The action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on

$$E[p^{\infty}] = \varprojlim_{n} E[p^{n}]$$

gives a continuous representation

$$ho_{\mathsf{E}} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) o \mathrm{GL}_2(\mathbb{Z}_{\mathsf{p}}).$$



We say that E has complex multiplication by an imaginary quadratic field K if

$$\operatorname{End}(E) = \operatorname{an order in } K$$

We say that E has complex multiplication by an imaginary quadratic field K if

$$\operatorname{End}(E) = \operatorname{an order in } K$$

If E has complex multiplication by K, then

$$\rho_E \cong \rho_E \otimes \chi_K$$

where χ_K is the quadratic character of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\overline{\mathbb{Q}}^{\ker \chi} = K$.

We say that E has complex multiplication by an imaginary quadratic field K if

$$\operatorname{End}(E) = \operatorname{an order in } K$$

If E has complex multiplication by K, then

$$\rho_E \cong \rho_E \otimes \chi_K$$

where χ_K is the quadratic character of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\overline{\mathbb{Q}}^{\ker \chi} = K$.

We say that E has complex multiplication by an imaginary quadratic field K if

$$\operatorname{End}(E) = \operatorname{an order in} K$$

If E has complex multiplication by K, then

$$\rho_E \cong \rho_E \otimes \chi_K$$

where χ_K is the quadratic character of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\overline{\mathbb{Q}}^{\ker \chi} = K$. This implies: up to conjugation,

$$\operatorname{Im} \rho_{\mathsf{E}} \subset \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$$

We say that E has complex multiplication by an imaginary quadratic field K if

$$\operatorname{End}(E) = \operatorname{an order in} K$$

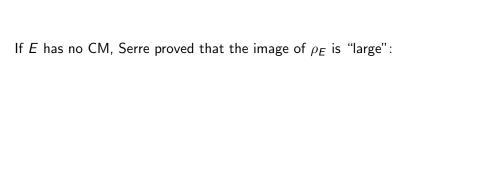
If E has complex multiplication by K, then

$$\rho_E \cong \rho_E \otimes \chi_K$$

where χ_K is the quadratic character of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\overline{\mathbb{Q}}^{\ker \chi} = K$. This implies: up to conjugation,

$$\operatorname{Im} \rho_{\mathsf{E}} \subset \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$$

The image is "small".



If *E* has no CM, Serre proved that the image of ρ_E is "large": up to conjugation,

$$\operatorname{Im}
ho_{\mathsf{E}}\supset egin{pmatrix} 1+p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \ p^n\mathbb{Z}_p & 1+p^n\mathbb{Z}_p \end{pmatrix}\cap\operatorname{SL}_2(\mathbb{Z}_p)$$

for some n, i.e., $\operatorname{Im} \rho_E$ contains a congruence subgroup of $\operatorname{SL}_2(\mathbb{Z}_p)$.

If E has no CM, Serre proved that the image of ρ_E is "large": up to conjugation,

$$\operatorname{Im}
ho_{\mathsf{E}}\supset egin{pmatrix} 1+p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \ p^n\mathbb{Z}_p & 1+p^n\mathbb{Z}_p \end{pmatrix}\cap\operatorname{SL}_2(\mathbb{Z}_p)$$

for some n, i.e., $\operatorname{Im} \rho_E$ contains a congruence subgroup of $\operatorname{SL}_2(\mathbb{Z}_p)$.

In other words: the size of the image of ρ_E detects exactly whether E is "special" or "generic".

Given

$$\rho \colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_n(A)$$

we can look at symmetries of the type

$$\rho \otimes \chi \cong \rho \circ \sigma$$

where σ is an automorphism of A.

Given

$$\rho \colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_n(A)$$

we can look at symmetries of the type

$$\rho \otimes \chi \cong \rho \circ \sigma$$

where σ is an automorphism of A.

We call (σ, χ) a *conjugate self-twist* of ρ .

Given

$$\rho \colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_n(A)$$

we can look at symmetries of the type

$$\rho \otimes \chi \cong \rho \circ \sigma$$

where σ is an automorphism of A.

We call (σ, χ) a *conjugate self-twist* of ρ .

Conjugate self-twists form a group Σ .

Easy: the size of ${\rm Im} \rho$ has a natural bound determined by all of its conjugate self-twists:	

Easy: the size of $\operatorname{Im} \rho$ has a natural bound determined by all of its conjugate self-twists:

 $A^{\Sigma} = \text{subring of } A \text{ of elements fixed by all conjugate self-twists}$

Easy: the size of ${\rm Im}\, \rho$ has a natural bound determined by all of its conjugate self-twists:

 $A^{\Sigma} =$ subring of A of elements fixed by all conjugate self-twists

(Usually:) There exists an open subgroup H of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$\rho(H) \subset \mathrm{GL}_2(A_0)$$

Easy: the size of ${\rm Im}\,\rho$ has a natural bound determined by all of its conjugate self-twists:

 $A^{\Sigma}=$ subring of A of elements fixed by all conjugate self-twists

(Usually:) There exists an open subgroup H of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$\rho(H) \subset \mathrm{GL}_2(A_0)$$

Question: Is $\operatorname{Im} \rho$ large in $\operatorname{GL}_2(A_0)$?

 $\textit{Easy:}\$ the size of $\operatorname{Im}\rho$ has a natural bound determined by all of its conjugate self-twists:

 $A^{\Sigma} =$ subring of A of elements fixed by all conjugate self-twists

(Usually:) There exists an open subgroup H of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$\rho(H) \subset \mathrm{GL}_2(A_0)$$

Question: Is $\operatorname{Im} \rho$ large in $\operatorname{GL}_2(A_0)$?

(Does the size of $\operatorname{Im} \rho$ detect precisely the symmetries of ρ ?)

For some of these we have a largeness result.

For some of these we have a largeness result. For instance:

Theorem (Momose 1981, Ribet 1985)

For a non-CM cuspidal modular form f and almost all p, the image of the p-adic Galois representation

$$\rho_{f,p} \colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathcal{O})$$

attached to f contains a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}^\Sigma)$.

For some of these we have a largeness result. For instance:

Theorem (Momose 1981, Ribet 1985)

For a non-CM cuspidal modular form f and almost all p, the image of the p-adic Galois representation

$$\rho_{f,p} \colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathcal{O})$$

attached to f contains a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}^\Sigma)$.

For some of these we have a largeness result. For instance:

Theorem (Momose 1981, Ribet 1985)

For a non-CM cuspidal modular form f and almost all p, the image of the p-adic Galois representation

$$\rho_{f,p} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$$

attached to f contains a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}^\Sigma)$.

There are also results for Hilbert and Siegel modular forms (Nekovar, Dieulefait–Zenteno), Hida and Coleman families of modular forms (Hida, J. Lang, C.–lovita–Tilouine), Siegel–Hida families (Hida–Tilouine).

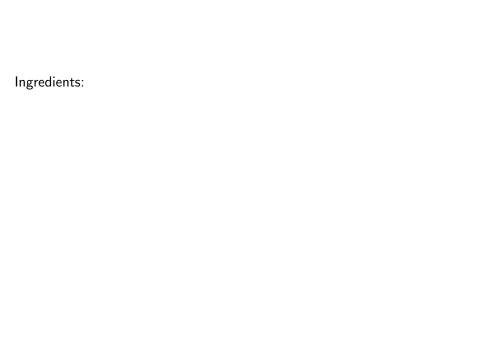
Bellaïche proves a purely algebraic result:

Theorem (Bellaïche 2017)

Consider a profinite group G, a local pro-p integral domain A and a continuous representation

$$\rho \colon G \to \mathrm{GL}_2(A)$$
.

Assume that ρ is irreducible, non-induced and "regular". Then there exists a subring A_0 of A such that the image of ρ contains a congruence subgroup of $\mathrm{SL}_2(A_0)$.



• replace representations by pseudorepresentations;

- replace representations by pseudorepresentations;
- $\ \ \, \ \, \ \,$ replace ${\rm GL}_2(A)$ by the units in a generalized matrix algebra (GMA);

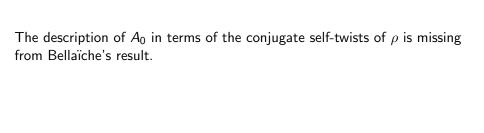
- replace representations by pseudorepresentations;
- $oldsymbol{\circ}$ replace $\operatorname{GL}_2(A)$ by the units in a generalized matrix algebra (GMA);
- generalize Pink's Lie algebra theory to GMAs.

- replace representations by pseudorepresentations;
- $oldsymbol{\circ}$ replace $\operatorname{GL}_2(A)$ by the units in a generalized matrix algebra (GMA);
- generalize Pink's Lie algebra theory to GMAs.

- replace representations by pseudorepresentations;
- $oldsymbol{0}$ replace $\mathrm{GL}_2(A)$ by the units in a generalized matrix algebra (GMA);
- 3 generalize Pink's Lie algebra theory to GMAs.
- 1 and 2 are important for treating reducible residual representations, and work in every dimension

Ingredients:

- replace representations by pseudorepresentations;
- $oldsymbol{@}$ replace $\mathrm{GL}_2(A)$ by the units in a generalized matrix algebra (GMA);
- 3 generalize Pink's Lie algebra theory to GMAs.
- $\boldsymbol{1}$ and $\boldsymbol{2}$ are important for treating reducible residual representations, and work in every dimension
- 3 only exists in dimension 2



The description of A_0 in terms of the conjugate self-twists of ρ is missing from Bellaïche's result.

Theorem (C.-Lang-Medvedovsky 2019)

Consider a profinite group G, a local pro-p integral domain A and a continuous representation

$$\rho \colon G \to \mathrm{GL}_2(A).$$

Assume that ρ is irreducible, non-induced and "regular". Then the image of ρ contains a congruence subgroup of $\mathrm{SL}_2(A^\Sigma)$.

The description of A_0 in terms of the conjugate self-twists of ρ is missing from Bellaïche's result.

Theorem (C.-Lang-Medvedovsky 2019)

Consider a profinite group G, a local pro-p integral domain A and a continuous representation

$$\rho \colon G \to \mathrm{GL}_2(A).$$

Assume that ρ is irreducible, non-induced and "regular". Then the image of ρ contains a congruence subgroup of $\mathrm{SL}_2(A^\Sigma)$.

The description of A_0 in terms of the conjugate self-twists of ρ is missing from Bellaïche's result.

Theorem (C.-Lang-Medvedovsky 2019)

Consider a profinite group G, a local pro-p integral domain A and a continuous representation

$$\rho \colon G \to \mathrm{GL}_2(A)$$
.

Assume that ρ is irreducible, non-induced and "regular". Then the image of ρ contains a congruence subgroup of $\mathrm{SL}_2(A^{\Sigma})$.

We can use this result to recover the known large image results.

with some tricks one can enlarge the residue field of Bellaïche's ring to get a new ring A'_0 ;

- with some tricks one can enlarge the residue field of Bellaïche's ring to get a new ring A'_0 ;
- ▶ conjugate self-twists of ρ can be lifted to conjugate self-twists of the universal deformation ring of $\overline{\rho}$;

- with some tricks one can enlarge the residue field of Bellaïche's ring to get a new ring A_0' ;
- conjugate self-twists of ρ can be lifted to conjugate self-twists of the universal deformation ring of $\overline{\rho}$;
- $ightharpoonup A^{\Sigma}$ contains A'_0 , it is finite as a A'_0 -module, and their fields of fractions coincide.

- with some tricks one can enlarge the residue field of Bellaïche's ring to get a new ring A_0' ;
- conjugate self-twists of ρ can be lifted to conjugate self-twists of the universal deformation ring of $\overline{\rho}$;
- $ightharpoonup A^{\Sigma}$ contains A'_0 , it is finite as a A'_0 -module, and their fields of fractions coincide.

- with some tricks one can enlarge the residue field of Bellaïche's ring to get a new ring A'_0 ;
- conjugate self-twists of ρ can be lifted to conjugate self-twists of the universal deformation ring of $\overline{\rho}$;
- ▶ A^{Σ} contains A'_0 , it is finite as a A'_0 -module, and their fields of fractions coincide. ($\Longrightarrow A^{\Sigma}$ -large = A'_0 -large)

- with some tricks one can enlarge the residue field of Bellaïche's ring to get a new ring A'_0 ;
- ightharpoonup conjugate self-twists of ho can be lifted to conjugate self-twists of the universal deformation ring of $\overline{
 ho}$;
- ▶ A^{Σ} contains A'_0 , it is finite as a A'_0 -module, and their fields of fractions coincide. ($\Longrightarrow A^{\Sigma}$ -large = A'_0 -large)

Questions:

- with some tricks one can enlarge the residue field of Bellaïche's ring to get a new ring A_0' ;
- ightharpoonup conjugate self-twists of ho can be lifted to conjugate self-twists of the universal deformation ring of $\overline{
 ho}$;
- ▶ A^{Σ} contains A'_0 , it is finite as a A'_0 -module, and their fields of fractions coincide. ($\Longrightarrow A^{\Sigma}$ -large = A'_0 -large)

Questions:

► Can this result be generalized to higher dimension?

- with some tricks one can enlarge the residue field of Bellaïche's ring to get a new ring A_0' ;
- ightharpoonup conjugate self-twists of ho can be lifted to conjugate self-twists of the universal deformation ring of $\overline{
 ho}$;
- ▶ A^{Σ} contains A'_0 , it is finite as a A'_0 -module, and their fields of fractions coincide. ($\Longrightarrow A^{\Sigma}$ -large = A'_0 -large)

Questions:

- ► Can this result be generalized to higher dimension?
- How can one characterize the level of the congruence subgroup contained in the image?

Thank you for your attention!