Finiteness and periodicity for β-continued fractions

joint work in progress with Zuzana Másáková and Tomáš Vávra

Francesco Veneziano

DIMA
Università di Genova

24 ottobre 2019

Continued Fractions

A continued fraction is an expression of the form

$$[a_0, a_1, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = \frac{p_n}{q_n}$$

Continued Fractions

A continued fraction is an expression of the form

$$[a_0, a_1, \dots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = \frac{p_n}{q_n}$$

In the classical setting, we take the a_i to be positive integers. In this case it make sense to consider an infinite sequence of a_i's and the corresponding limit of the values p_n/q_n.

Continued Fraction Expansion

Starting from a real number $\alpha = \alpha_0$ we define the iteration

$$a_n = \lfloor \alpha_n \rfloor$$
$$\alpha_{n+1} = (\alpha_n - a_n)^{-1}$$

Continued Fraction Expansion

Starting from a real number $\alpha = \alpha_0$ we define the iteration

$$a_n = \lfloor \alpha_n \rfloor$$
$$\alpha_{n+1} = (\alpha_n - a_n)^{-1}$$

and the recurrences

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad p_{-1} = 1, p_{-2} = 0,$$
$$q_n = a_n q_{n-1} + q_{n-2}, \qquad q_{-1} = 0, q_{-2} = 1.$$

Continued Fraction Expansion

Starting from a real number $\alpha = \alpha_0$ we define the iteration

$$a_n = \lfloor \alpha_n \rfloor$$
$$\alpha_{n+1} = (\alpha_n - a_n)^{-1}$$

and the recurrences

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad p_{-1} = 1, p_{-2} = 0,$$
$$q_n = a_n q_{n-1} + q_{n-2}, \qquad q_{-1} = 0, q_{-2} = 1.$$

The a_n are called *partial quotients*
The α_n are called *complete quotients*
The p_n/q_n are called *convergents*

Classical results

$$\lim_{n\to\infty}[a_0, \ldots, a_n] = \lim_{n\to\infty} p_n/q_n = \alpha$$

Classical results

$$\lim_{n \to \infty} [a_0, \ldots, a_n] = \lim_{n \to \infty} p_n / q_n = \alpha$$

The expansion is essentially unique

Classical results

$$\lim_{n\to\infty} [a_0, \dots, a_n] = \lim_{n\to\infty} p_n/q_n = \alpha$$

The expansion is essentially unique
The iteration stops if and only if α is rational

Classical results

$$\lim_{n\to\infty} [a_0, \ldots, a_n] = \lim_{n\to\infty} p_n/q_n = \alpha$$

The expansion is essentially unique
The iteration stops if and only if α is rational
The expansion is eventually periodic if and only if α is a quadratic irrational (Lagrange)

Classical results

$$\lim_{n\to\infty}[a_0,\dots,a_n]=\lim_{n\to\infty}p_n/q_n=\alpha$$

The expansion is essentially unique

The iteration stops if and only if α is rational

The expansion is eventually periodic if and only if α is a quadratic irrational (Lagrange)

The convergents provide very good rational approximations to α.

A Question of Rosen

Question (Rosen '77)

Is it possible to devise a continued fraction that represents uniquely all real numbers, so that the finite continued fractions represent the elements of an algebraic number field, and conversely, every element of the number field is represented by a finite continued fraction?

A Question of Rosen

Question (Rosen '77)

Is it possible to devise a continued fraction that represents uniquely all real numbers, so that the finite continued fractions represent the elements of an algebraic number field, and conversely, every element of the number field is represented by a finite continued fraction?

Rosen gives one example of such a construction for the field $\mathbb{Q}(\sqrt{5})$ and partial quotients which are integral multiples of $\phi = \frac{1+\sqrt{5}}{2}$.

A Question of Rosen

Question (Rosen '77)

Is it possible to devise a continued fraction that represents uniquely all real numbers, so that the finite continued fractions represent the elements of an algebraic number field, and conversely, every element of the number field is represented by a finite continued fraction?

Rosen gives one example of such a construction for the field $\mathbb{Q}(\sqrt{5})$ and partial quotients which are integral multiples of $\phi = \frac{1+\sqrt{5}}{2}$.

Bernat '06 gives a different construction again for $\mathbb{Q}(\sqrt{5})$.

β-expansions

Let $\beta > 1$ be an algebraic integer. Any real number x can be expanded in base-β as

$$x = \pm \sum_{i=-\infty}^{k} x_i \beta^i.$$

The digits x_i belong to the set $\{0, 1, \ldots, \lceil \beta \rceil - 1\}$, and are selected according to a greedy algorithm.

β-expansions

Let $\beta > 1$ be an algebraic integer. Any real number x can be expanded in base-β as

$$x = \pm \sum_{i=-\infty}^{k} x_i \beta^i.$$

The digits x_i belong to the set $\{0, 1, \ldots, \lceil \beta \rceil - 1\}$, and are selected according to a greedy algorithm.
Not all expansions are admissible. $\phi^2 = \phi + 1$

β-integers

Consider the set $\mathbb{Z}_\beta$ of the real numbers whose β-expansion uses only non-negative powers of β. These numbers are called β-integers.

β-integers

Consider the set $\mathbb{Z}_\beta$ of the real numbers whose β-expansion uses only non-negative powers of β. These numbers are called β-integers.
They form a discrete subset of the algebraic integers in the field $\mathbb{Q}(\beta)$.

β-integers

Consider the set $\mathbb{Z}_\beta$ of the real numbers whose β-expansion uses only non-negative powers of β. These numbers are called β-integers.
They form a discrete subset of the algebraic integers in the field $\mathbb{Q}(\beta)$.
For some special β's e.g. for Pisot numbers, it it possible to give an algebraic characterization of this set in terms of their algebraic conjugates.

β-fractionary expansion

For a positive real number x, define

$$\lfloor x \rfloor_\beta = \max\{a \in \mathbb{Z}_\beta \mid a \le x\}.$$

Replace $\lfloor \cdot \rfloor$ by $\lfloor \cdot \rfloor_\beta$ in the definition of the continued fraction expansion.

β-fractionary expansion

For a positive real number x, define

$$\lfloor x \rfloor_\beta = \max\{a \in \mathbb{Z}_\beta \mid a \leq x\}.$$

Replace $\lfloor \cdot \rfloor$ by $\lfloor \cdot \rfloor_\beta$ in the definition of the continued fraction expansion.
Bernat proved that for $\beta = \phi$ this construction solves Rosen's problem for the field $\mathbb{Q}(\sqrt{5})$.

β-fractionary expansion

For a positive real number x, define

$$\lfloor x \rfloor_\beta = \max\{a \in \mathbb{Z}_\beta \mid a \leq x\}.$$

Replace $\lfloor \cdot \rfloor$ by $\lfloor \cdot \rfloor_\beta$ in the definition of the continued fraction expansion.
Bernat proved that for $\beta = \phi$ this construction solves Rosen's problem for the field $\mathbb{Q}(\sqrt{5})$. The proof is intricate, it uses that ϕ is a quadratic Pisot number smaller than 2.

β-fractionary expansion

For a positive real number x, define

$$\lfloor x \rfloor_\beta = \max\{a \in \mathbb{Z}_\beta \mid a \leq x\}.$$

Replace $\lfloor \cdot \rfloor$ by $\lfloor \cdot \rfloor_\beta$ in the definition of the continued fraction expansion.

Bernat proved that for $\beta = \phi$ this construction solves Rosen's problem for the field $\mathbb{Q}(\sqrt{5})$. The proof is intricate, it uses that ϕ is a quadratic Pisot number smaller than 2.

Question (Bernat '06)

For which other numbers (quadratic Pisot) does the same conclusion hold?

Periodicity and finiteness for the β-fractionary expansion

Let $\beta > 1$ be an algebraic integer.

(CFP)

We say that β has the (CFP) property if the β-fractionary expansion of every element of $\mathbb{Q}(\beta)$ if finite or eventually periodic.

Periodicity and finiteness for the β-fractionary expansion

Let $\beta > 1$ be an algebraic integer.

(CFP)

We say that β has the (CFP) property if the β-fractionary expansion of every element of $\mathbb{Q}(\beta)$ if finite or eventually periodic.

(CFF)

We say that β has the (CFF) property if the β-fractionary expansion of every element of $\mathbb{Q}(\beta)$ if finite.

Results on quadratic β's

From now on, let $\beta > 1$ be a quadratic integer, and let β' be its algebraic conjugate.

Theorem (Másáková,V,Vávra)

If $|\beta'| < \beta$ *(Perron numbers), then* (CFP) *holds.*
Every purely periodic element in $\mathbb{Q}(\beta)$ *has partial quotients in* $\{1, \ldots, \lfloor \beta \rfloor\}$.

Results on quadratic β's

From now on, let $\beta > 1$ be a quadratic integer, and let β' be its algebraic conjugate.

Theorem (Másáková,V,Vávra)

If $|\beta'| < \beta$ *(Perron numbers), then* (CFP) *holds.*
Every purely periodic element in $\mathbb{Q}(\beta)$ *has partial quotients in* $\{1, \ldots, \lfloor\beta\rfloor\}$.

We use an argument of diophantine approximation and a comparison lemma to estimate the relative growhts of the sequences p_n, q_n and their conjugates.

Theorem (Másáková,V,Vávra)

The four Perron numbers

$$\frac{1+\sqrt{5}}{2}, \qquad 1+\sqrt{2}, \qquad \frac{1+\sqrt{13}}{2}, \qquad \frac{1+\sqrt{17}}{2}$$

have (CFF)*, and are the only quadratic Perron numbers smaller than 3 with property* (CFF)*.*

Theorem (Másáková,V,Vávra)

The four Perron numbers

$$\frac{1+\sqrt{5}}{2}, \qquad 1+\sqrt{2}, \qquad \frac{1+\sqrt{13}}{2}, \qquad \frac{1+\sqrt{17}}{2}$$

have (CFF), *and are the only quadratic Perron numbers smaller than 3 with property* (CFF).

We apply the previous theorem and argue about admissible sequences of partial quotients.

Theorem (Másáková,V,Vávra)

The four Perron numbers

$$\frac{1+\sqrt{5}}{2}, \qquad 1+\sqrt{2}, \qquad \frac{1+\sqrt{13}}{2}, \qquad \frac{1+\sqrt{17}}{2}$$

have (CFF)*, and are the only quadratic Perron numbers smaller than 3 with property* (CFF)*.*

We apply the previous theorem and argue about admissible sequences of partial quotients.

$$\frac{164+65\sqrt{17}}{251} = [\overline{1,1,2,1,1,2,2,2,2}]$$

$$\frac{164+65\sqrt{17}}{251} = [1,1,\beta,2\beta^3+\beta^2+1,\beta^3+\beta+1,2,\beta+1]$$

Conjecture (McMullen '08)

Every real quadratic number field contains infinitely many elements whose (classical) continued fraction expansion consists only in 1's and 2's.

Conjecture (McMullen '08)

Every real quadratic number field contains infinitely many elements whose (classical) continued fraction expansion consists only in 1's and 2's.

Conjecture (Mercat '13)

Every real quadratic number field contains one element whose (classical) continued fraction expansion consists only in 1's and 2's.

Under Mercat's conjecture, no quadratic $\beta > 3$ can have property (CFF).

Theorem (Másáková,V,Vávra)

If $\beta' > \beta$, *then* (CFP) *never holds.*
Every purely periodic elements in $\mathbb{Q}(\beta)$ *has partial quotients in* $\{1, \dots, \lfloor \beta \rfloor\}$.

Theorem (Másúková,V,Vávra)

If $\beta' > \beta$, then (CFP) *never holds.*
Every purely periodic elements in $\mathbb{Q}(\beta)$ has partial quotients in $\{1, \ldots, \lfloor\beta\rfloor\}$.

We use an argument of algebraic number theory and a characterization of pure periodicity.