On interesting subsequences of the sequence of primes

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Abstract

It is a famous result that the set of quotients of prime numbers is dense in the set of positive real numbers. It is a motivation to wide study of denseness properties of subsets of positive integers on real half-line, see e.g. [1, 11, 12, 14]. One can meet it as an exercise on course of number theory, see [2, Problem 218], [3, Ex. 4.19], [9, Ex. 7, p. 107], [10, Thm. 4] and also in several articles, e.g. [4, Cor. 4], [6, Thm. 4], [13, Cor. 2] (according to the last reference, the result was known to Sierpiski, who credits it to Schinzel [8]). The authors of [4] generalized this result to the subsets of prime numbers in given arithmetic progressions.

Motivated by the article [5] on "light" subsets of positive integers (i.e. subsets with slowly growing counting fuctions) we focus on the family of subsets $\mathcal{P}_k = \{p_1^{(k)} < p_2^{(k)} < p_3^{(k)} < ...\}$, $k \in \mathbb{N}$, of prime numbers such that every next set contains these elements of the preceding one indexed by prime numbers. As a consequence, every next set is a zero asymptotic density subset of the preceding one. Although the sets $\mathcal{P}_k$ are "lighter and lighter" as k increases, we will show that all of them have dense quotient sets in the set of positive real numbers. We will also study the sets $\mathcal{P}_n^T = \{p_n^{(k)} : k \in \mathbb{N}\}$, $n \in \mathbb{N}$, and $\mathrm{Diag}\mathcal{P} = \{p_k^{(k)} : k \in \mathbb{N}\}$. We will prove that, in the opposition to the sets $\mathcal{P}_k$, their quotient sets are not dense in $\mathbb{R}^+$.

This is a joint work with János T. Tóth.

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