DOCTORAL PROGRAM DISCRETE MATHEMATICS
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# Structural Properties of Weakly Directed Families with Applications to Factorization Theory

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# Outline

A successful marriage

Sets of lengths and their structure

Additive models

# Factorization Theory (FT)

Similar to the integers, non-zero non-unit elements in many (classes of unital) rings can be expressed as (finite) products of irreducible elements, but unlike the integers, these factorizations need not be unique

The basic goal of FT is to study phenomena stemming from the uniqueness of these factorizations and classify them by an assortment of (arithmetical, combinatorial, algebraic, topological, ...) invariants

The theory had its origins in algebraic NT, and was later extended to commutative rings and commutative monoids, with an emphasis on integral domains and cancellative, commutative monoids

# Short history of FT

Only in recent years, people have started considering more abstract settings, especially domains and (possibly) non-commutative monoids

- *Multiplicative ideal theory*: Krull, E. Noether, & Prüfer (1930s-1940s)
- *Narkiewicz* (1970s): Does the arithmetic characterize the class group?
- *D.D. Anderson & al.* (1980s): Integral domains and rings with zero-divisors
- *Chapman & Smith* (1993; 1998): Dedekind domains
- *Halter-Koch & Geroldinger* (1990s): Cancellative, commutative monoids
- *Halter-Koch* (1997): Transfer homomorphisms
- *Halter-Koch & Geroldinger* (2006): "Non-Unique Factorizations" is published
- *Baeth & Smertnig* (2015): Non-commutative, cancellative monoids/categories
- *Yu. Fan & T.* (Oct 2016): Arbitrary monoids

In all of the above cases, sets of lengths and other widgets derived from them have been among the most studied invariants in FT

# Arithmetic Combinatorics (AC)

On the other hand, it is hard to provide a concise description of AC: The field has been expanding very quickly in the last two decades, and its identity is, to a large extent, still undetermined

Most relevant for this talk: The study of sumsets over the integers, with an emphasis on structural properties under extremal conditions

Recall that, if $X_1, \ldots, X_n \subseteq \mathbf{Z}$, then

$$X_1 + \cdots + X_n := \{x_1 + \cdots + x_n : x_1 \in X_1, \ldots, x_n \in X_n\} \subseteq \mathbf{Z}$$

is the sumset of the $n$-tuple $(X_1, \ldots, X_n)$

**Example**: It is trivial that, if $X, Y \subseteq \mathbf{Z}$ are non-empty, then

$$|X + Y| \geq |X| + |Y| - 1,$$

and if $2 \leq |X|, |Y| < \infty$ and equality is attained, then $X$ and $Y$ are APs (arithmetic progressions) with the same difference

# Overlap between the two fields

It has been known for a long time that FT and AC have strong ties:

1. Zero-sum and product-one sequences: The arithmetic of Krull domains (e.g., Dedekind domains) and C-domain (e.g., orders in number fields) can be fully understood by reduction to monoids of zero-sum sequences (over abelian groups) and product-one sequences (over arbitrary groups), resp. This results into deep connections with extremal questions in AC (related, e.g., to the EGZ constant, the Davenport constant, the Harborth constant, etc.)
2. Addition theorems (Kneser, Kemperman-Scherk, Cauchy-Davenport type inequalities, ...): A key tool in the study of the monoids of point 1, and to investigate the structure of sets of lengths (see later)
3. Power (set) monoids: The subject of my talk last year
4. Weakly directed (WF) families: The subject of my talk this year

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# Sets of lengths

$H = (H, \cdot)$: multiplicatively written monoid with identity $1_H$

$H^\times$: the group of units of $H$ [$a \in H^\times$ iff $ab = ba = 1_H$ for some $b \in H$]

$\mathcal{A}(H)$: the set of atoms (or irreducible elements) of $H$ [$u \in \mathcal{A}(H)$ iff $u \notin H^\times$ and $u = ab$ for some $a, b \in H$ implies $a \in H^\times$ or $b \in H^\times$]

Given $x \in H$, we set $\mathsf{L}_H(x) := \{0\} \subseteq \mathbf{N}$ if $x = 1_H$, otherwise

$$\mathsf{L}_H(x) := \{k \in \mathbf{N}^+ : x = u_1 \cdots u_k \text{ for some } u_1, \ldots, u_k \in \mathcal{A}(H)\} \subseteq \mathbf{N}^+$$

We call $\mathsf{L}_H(x)$ the set of lengths of $x$, and we let

$$\mathcal{L}(H) := \{\mathsf{L}_H(x) : x \in H\} \setminus \{\emptyset\} \subseteq \mathcal{P}(\mathbf{N})$$

be the system of sets of lengths of $H$

# Unions of sets of lengths, distances, and elasticity

For every $k \in \mathbf{N}$, we define

$$\mathcal{U}_k(H) := \bigcup \{L \in \mathcal{L}(H) : k \in L\}$$

Given $L \subseteq \mathbf{N}$ and $d \in \mathbf{N}^+$, we say that $d$ is a distance of $L$ if

$$L \cap [\![l, l+d]\!] = \{l, l+d\} \quad \text{for some } l \in \mathbf{N},$$

and we denote by $\Delta(L)$ the set of distances of $L$. Accordingly,

$$\Delta(H) := \bigcup_{L \in \mathcal{L}(H)} \Delta(L) \quad \text{and} \quad \delta(H) := \begin{cases} 1 & \text{if } \Delta(H) = \emptyset \\ \min \Delta(H) & \text{otherwise} \end{cases}$$

We call $\Delta(H)$ the set of distances of $H$

# Basic goals and results

One of the main goals of FT is to "fully understand" the structure of sets of lengths, and of invariants derived from them (in particular, unions), to the extent of proving realization theorems (within a prescribed class)

In particular, it is known that, for the ring of integers of a number field, the unions are intervals (the smoothest thing we may ask for!)

More in general, We. Gao & Geroldinger (2009) proved that a Krull monoid $H$ with finite class group satisfies the Structure Theorem for Unions (STU), i.e., there exist $M \in \mathbf{N}$ and $d \in \mathbf{N}^+$ s.t., for all large $k$,

- $\mathcal{U}_k(H) \subseteq d + k \cdot \mathbf{Z}$, and
- $\mathcal{U}_k(H) \cap [\![\inf \mathcal{U}_k(H) + M, \sup \mathcal{U}_k(H) - M]\!]$ is an AP

[It is easy to prove that, if a monoid $H$ satisfies the STU, then $d = \delta(H)$]

**Questions**: Is this the best we can hope for in general? Or is there even more structure that still has to be revealed? **Answers:** No and yes, resp.

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# Weakly directed families

Let $\mathcal{L} \subseteq \mathcal{P}(\mathbf{N}) \setminus \{\emptyset\}$. We say that $\mathcal{L}$ is a weakly directed (WD) family if, for all $L_1, L_2 \in \mathcal{L}$, there is $L \in \mathcal{L}$ with $L_1 + L_2 \subseteq L$

WD families provide an effective "additive model" for answering questions on the structure of sets of lengths and their unions (we may say that WD families capture the "combinatorial essence" of sets of lengths)

**How?** The basic point is that, if $H$ is a monoid, then

$$\mathsf{L}_H(x) + \mathsf{L}_H(y) \subseteq \mathsf{L}_H(xy), \quad \text{for all } x, y \in H,$$

which implies that the system of sets of lengths of $H$ is a WF family

# Beyond the Structure Theorem

Let $\mathcal{L} \subseteq \mathcal{P}(\mathbf{N})$ and, for each $k \in \mathbf{N}$, set

$$\mathcal{U}_k := \bigcup \{L \in \mathcal{L} : k \in L\} \subseteq \mathbf{N}$$

and, for easiness of notation,

$$\lambda_k := \inf \mathcal{U}_k(\mathcal{L}) \quad \text{and} \quad \rho_k := \sup \mathcal{U}_k$$

We say that $\mathcal{L}$ satisfies:

- the STU if (...you guess the rest...)
- the Strong Structure Theorem for Unions (SSTU) if there are $d, \mu \in \mathbf{N}^+$ and finite sets $\mathcal{U}'_0, \mathcal{U}''_0, \ldots, \mathcal{U}'_{\mu-1}, \mathcal{U}''_{\mu-1} \subseteq \mathbf{N}$ s.t.

$$\mathcal{U}_k = (\lambda_k + \mathcal{U}'_{k \bmod \mu}) \uplus \mathcal{P}_k \uplus (\rho_k - \mathcal{U}''_{k \bmod \mu}) \subseteq k + d \cdot \mathbf{Z},$$

  for all large $k$, where $\mathcal{P}_k$ is an AP with difference $d$

If $\mathcal{L}$ is WD and satisfies the STU, then $d = \delta(\mathcal{L})$, where $\delta(\mathcal{L})$ is (...what you are thinking it is...), and SSTU $\implies$ STU

# The importance of being elastic

**Question**: Does (the system of sets of lengths of) a Krull domain with finite class group satisfy the SSTU?

**Answer**: Yes, and so do many more domains and *cancellative* monoids

How? Given $\mathcal{L} \subseteq \mathcal{P}(\mathbf{N})$, define the elasticity of $\mathcal{L}$ by

$$\rho(\mathcal{L}) := \sup\{\rho(L) : L \in \mathcal{L}\} \in \{0\} \cup [1, \infty],$$

where for every $L \subseteq \mathbf{N}$ we set

$$\rho(L) := \begin{cases} 0 & \text{if } L^+ := L \cap \mathbf{N}^+ = \emptyset \\ \sup L / \min L^+ & \text{otherwise} \end{cases}$$

$\mathcal{L}$ is said to have accepted elasticity if $\rho(\mathcal{L}) = \rho(L) < \infty$ for some $L \in \mathcal{L}$

# Main result

## Theorem (T., July 2017)

*Every WD family with accepted elasticity satisfies the SSTU*

The theorem is *very* general. It applies, e.g., to (the system of sets of lengths of) the following monoids or domains:

- numerical monoids (extremely relevant to invariant theory)
- Krull monoids of finite type (e.g., Krull monoids with finite class group), either commutative or not
- a variety of weakly Krull commutative domains (including all orders in number fields with finite elasticity)
- certain maximal orders in central simple algebras over global fields
- all monoids that are essentially equimorphic to a cancellative, commutative monoid $H$ s.t. $H/H^{\times}$ is finitely generated

# Work in progress

**Question.** Is it possible to extend the theorem to (unital) rings that need not be domains and, more in general, to non-cancellative monoids?

**Answer.** Yes! Besides the monoids mentioned in the previous slide:

- all local arithmetical congruence monoids, that is, submonoids of $(\mathbf{N}^+, \cdot)$ of the form $\{1\} \cup \{a + bk : k \in \mathbf{N}\}$ s.t. $a, b \in \mathbf{N}^+$ and $\gcd(a, b)$ is a prime power (of course, we need $a^2 \equiv a \bmod b$)
- all unit-cancellative, commutative monoids $H$ s.t. $H/H^\times$ has finitely many atoms that are not prime, satisfy the SSTU (monoids of modules and monoids of ideals are leading examples)

**Open question.** Does the SSTU hold for *all* orders in number fields? The answer leads to consider simple C-monoids as a canonical model

# Thanks for your attention!