An effective Chebotarev theorem under GRH

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Torino
26–27 Ottobre 2017

Chebotarev theorem: a review

Let $\mathbb{K} \hookrightarrow \mathbb{L}$ be a finite–Galois extension of number fields, with $G := \mathrm{Gal}(\mathbb{L}, \mathbb{K})$. Let $\mathcal{O}_\mathbb{K}$, $\mathcal{O}_\mathbb{L}$ be the corresponding rings of integral elements.
Fix a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_\mathbb{K}$ not ramifying in $\mathbb{L}$. For each prime ideal $\mathfrak{P}$ dividing $\mathfrak{p}\mathcal{O}_\mathbb{L}$, the Artin symbol $\left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{P}}\right]$ denotes the Frobenius automorphism corresponding to $\mathfrak{P}/\mathfrak{p}$, i.e. the element in G such that

$$\left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{P}}\right](\alpha) = \alpha^{\mathrm{N}\mathfrak{p}} \pmod{\mathfrak{P}} \qquad \forall \alpha \in \mathcal{O}_\mathbb{L}.$$

The set $\{\left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{P}}\right] : \mathfrak{P}|\mathfrak{p}\}$ is a conjugation class in G, also denoted $\left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right]$.

Conjecture (Frobenius) *For every conjugation class C, there are infinitely many $\mathfrak{p}$ with $\left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right] = C$; the Artin symbols equidistribute (in some sense) proportionally to $|C|/|G|$.*

Theorem (Chebotarev) *For every conjugation class C, the set*

$$\{\mathfrak{p} \colon \left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right] = C\} \quad \text{has Dirichlet density and it is} \quad = |C|/|G|.$$

Chebotarev's theorem (PNT-like) *For every conjugation class C*

$$|\{\mathfrak{p} \colon \left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right] = C, \mathrm{N}\mathfrak{p} \le x\}| \sim \frac{|C|}{|G|}\frac{x}{\log x} \quad x \to \infty.$$

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Conditional explicit bounds

Let ϵ_C be the characteristic function of the elements in C, and let

$$\psi_C(x) := \sum_{\mathrm{N}\mathfrak{I}\le x} \epsilon_C(\left[\tfrac{\mathbb{L}/\mathbb{K}}{\mathfrak{I}}\right])\Lambda_{\mathbb{K}}(\mathfrak{I}).$$

Theorem (Lagarias-Odlyzko '76) *(GRH) There exist absolute constants c_1, c_2 such that*

$$\Big|\frac{|G|}{|C|}\psi_C(x) - x\Big| \le \sqrt{x}(c_1 \log \Delta_{\mathbb{L}} \log x + c_2 n_{\mathbb{L}} \log^2 x).$$

Theorem (Oesterlé '79) *(GRH) For $x \ge 1$*

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Theorem (Schoenfeld '76) *(RH) For $x \ge 73$*

$$|\psi(x) - x| \le \frac{1}{8\pi}\sqrt{x}\log^2 x.$$

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general

Some tools: Include the ramifying prime ideals and Artin/Hecke

$$\theta(C;\mathfrak{p}^m) := \frac{1}{|I|}\sum_{a\in I}\epsilon_C(\tau^m a)$$

(I is the *inertia group* and τ is any of Frobenius automorphisms corresponding to $\mathfrak{P}/\mathfrak{p}$).

$$\psi(C;x) := \sum_{\mathrm{N}\mathfrak{I}\le x}\theta(C;\mathfrak{I})\Lambda_{\mathbb{K}}(\mathfrak{I}).$$

Write $\theta(C;\cdot)$ as linear combinations of irreducible characters for G:

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$$\sum_{\mathfrak{I}}\theta(C;\mathfrak{I})\Lambda_{\mathbb{K}}(\mathfrak{I})(\mathrm{N}\mathfrak{I})^{-s} = -\frac{|C|}{|G|}\sum_{\phi\in\hat{G}}\bar{\phi}(g)\frac{L'}{L}(s,\phi,\mathbb{L}/\mathbb{K}) \qquad \text{(Artin } L\text{-functions)}$$

Since C is fixed and $H := \langle g\rangle$ is cyclic, we have also

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where $\mathbb{E} := \mathbb{L}^H$ and each $L(s,\chi,\mathbb{L}/\mathbb{E})$ is abelian Artin, hence Hecke L-function (by class field theory).

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$$\psi^{(1)}(C;x) := \int_0^x \psi(C;u)\,\mathrm{d}u = \sum_{\mathrm{N}\mathfrak{I}\leq x} (x - \mathrm{N}\mathfrak{I})\theta(C;\mathfrak{I})\Lambda_{\mathbb{K}}(\mathfrak{I}).$$

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$$\psi^{(1)}(C;x) = -\frac{|C|}{|G|}\sum_{\chi}\bar{\chi}(g)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s,\chi,\mathbb{L}/\mathbb{E})\frac{x^{s+1}}{s(s+1)}\,\mathrm{d}s \qquad \forall x \geq 1,$$

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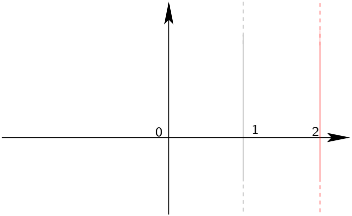

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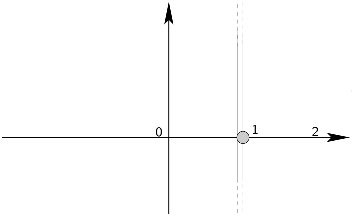

the pole in 1 gives the main term.

$$\frac{|G|}{|C|}\psi^{(1)}(C;x) = \frac{x^2}{2}$$

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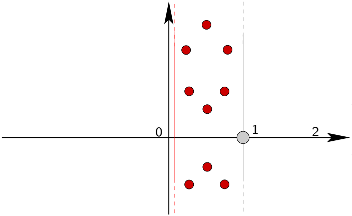

ρ runs on the set of zeros for all $L(s,\chi,\mathbb{L}/\mathbb{E})$; since $\prod_\chi L(s,\chi,\mathbb{L}/\mathbb{E}) = \zeta_{\mathbb{L}}(s)$, this is the set of zeros for the Dedekind of $\mathbb{L}$. Weights ϵ satisfy $|\epsilon(\rho)| = 1$.

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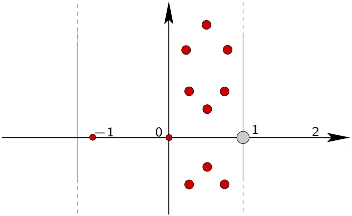

zeros in 0 and -1 produce special terms because they are poles for the kernel.

$$\frac{|G|}{|C|}\psi^{(1)}(C;x) = \frac{x^2}{2} - \sum_{\rho}\epsilon(\rho)\frac{x^{\rho+1}}{\rho(\rho+1)} - x r_C + r'_C$$

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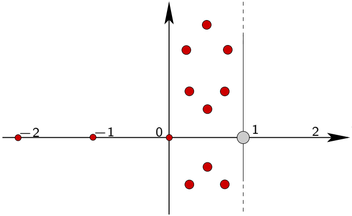

All other zeros produce an explicit term of size $x\log x$ which is independent of the discriminant.

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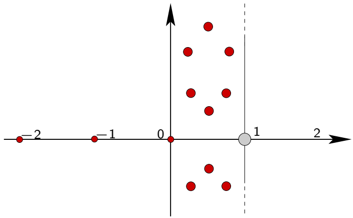

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Some tools: positive coefficients

Since $\psi^{(1)}(C;x) = \sum_{\mathrm{N}\mathfrak{I}\leq x}(x - \mathrm{N}\mathfrak{I})\theta(C;\mathfrak{I})\Lambda_{\mathbb{K}}(\mathfrak{I})$ and $\theta(C;\mathfrak{I})$ is nonnegative, one has

$$\psi(C;x) \leq [\psi^{(1)}(C;x+h) - \psi^{(1)}(C;x)]/h \qquad \text{as } h > 0,$$
$$\psi(C;x) \geq [\psi^{(1)}(C;x+h) - \psi^{(1)}(C;x)]/h \qquad \text{as } -x < h < 0.$$

So, we can recover bounds for $\psi(C;x)$ from analogous bounds for $\psi^{(1)}(C;x)$. This is a good idea since the results for $\psi^{(1)}(C;x)$ are stronger.

This approach needs bounds for several "non-trivial" objects. For example

$$\Big|\sum_{\rho}\epsilon(\rho)\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|$$

(ρ's are the non trivial zeros for $\zeta_{\mathbb{L}}$) which is split as

$$\leq \sum_{|\gamma|\leq T}\Big|\frac{x^{\rho}}{\rho}\Big| + \sum_{|\gamma|\leq T}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}-h(\rho+1)x^{\rho}}{h\rho(\rho+1)}\Big| + \sum_{|\gamma|>T}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|$$

$$\leq \sum_{|\gamma|\leq T}\frac{\sqrt{x}}{|\rho|} + \frac{|h|}{\sqrt{x}}\sum_{|\gamma|\leq T}1 + \Big(2\frac{x^{3/2}}{|h|} + o(1)\Big)\sum_{|\gamma|>T}\frac{1}{|\rho|^2}$$

and we have developed some tools producing goods bounds.

Some tools: positive coefficients

Since $\psi^{(1)}(C;x) = \sum_{\mathrm{N}\mathfrak{I}\le x}(x-\mathrm{N}\mathfrak{I})\theta(C;\mathfrak{I})\Lambda_{\mathbb{K}}(\mathfrak{I})$ and $\theta(C;\mathfrak{I})$ is nonnegative, one has

$$\begin{aligned}\psi(C;x) &\le [\psi^{(1)}(C;x+h)-\psi^{(1)}(C;x)]/h && \text{as } h>0,\\ \psi(C;x) &\ge [\psi^{(1)}(C;x+h)-\psi^{(1)}(C;x)]/h && \text{as } -x<h<0.\end{aligned}$$

So, we can recover bounds for $\psi(C;x)$ from analogous bounds for $\psi^{(1)}(C;x)$. This is a good idea since the results for $\psi^{(1)}(C;x)$ are stronger.

This approach needs bounds for several "non-trivial" objects. For example

$$\Big|\sum_{\rho}\epsilon(\rho)\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|$$

(ρ's are the non trivial zeros for $\zeta_{\mathbb{L}}$) which is split as

$$\le \sum_{|\gamma|\le T}\Big|\frac{x^\rho}{\rho}\Big| + \sum_{|\gamma|\le T}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}-h(\rho+1)x^\rho}{h\rho(\rho+1)}\Big| + \sum_{|\gamma|>T}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|$$

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and we have developed some tools producing goods bounds.

The strategy produces the following intermediate result.

Theorem (Grenié-M.) *(GRH) Let $x \geq 4$ and $T \geq 2\pi$ then:*

$$|\frac{|G|}{|C|}\psi(C;x) - x| \leq \sqrt{x}\left[F(x,T)\log\Delta_{\mathbb{L}} + G(x,T)n_{\mathbb{L}} + H(x,T)\right]$$

$$F(x,T) = \frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big) + \cdots, \qquad G(x,T) = \frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big) + \cdots,$$

$$H(x,T) = \frac{\sqrt{x}}{T} + \cdots.$$

Setting $T \approx_{\mathbb{L}} \sqrt{x}$ (well, actually a lot of computations need here, to control the secondary terms which depend on $\mathbb{K}$, $\mathbb{L}$, G and the class C) we reach a result similar to Back but for $\psi(C;x)$ (hence using $\theta(C)$, which is a smoothed version of the characteristic function for C), which is related to $\psi_C(x)$ (hence using the characteristic function of C) by

$$\psi(C;x) = \psi_C(x) + \text{ramification term}.$$

The ramification term is positive, hence the upper bound for $\psi(C;x)$ implies the same upper bound for $\psi_C(x)$. For lower bounds some difficult tricks are need, in order to reach the same conclusion for $\psi_C(x)$ and for $\psi(C;x)$.

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The ramifications term

$$\text{ramification term} \leq \min\left(\frac{|C|}{p}, 1\right) n_{\mathbb{K}}\, \mathfrak{n} \log x$$

where p is the smallest prime divisor of $|G|$, and $\mathfrak{n} := \frac{1}{n_{\mathbb{K}}} \sum_{\mathfrak{p} | \Delta_{\mathbb{L}/\mathbb{K}}} 1$.

$$\mathfrak{n} \leq \begin{cases} \frac{\log(\Delta_{\mathbb{L}}^{1/n_{\mathbb{K}}})}{\log 4} & \text{for all fields but } \mathbb{L} = \mathbb{Q}[\pm\sqrt{3}],\ \mathbb{Q}[\pm\sqrt{15}], \\ \frac{\log(\Delta_{\mathbb{L}}^{1/n_{\mathbb{K}}})}{\log 49} & \text{if } n_{\mathbb{L}} = 3, \\ \frac{\log(\Delta_{\mathbb{L}}^{1/n_{\mathbb{K}}})}{\log 22} & \text{if } |G| \text{ is not a prime (with 25 exceptions)}, \\ \frac{\log(\Delta_{\mathbb{L}}^{1/n_{\mathbb{K}}})}{\log\log(\Delta_{\mathbb{L}}^{1/n_{\mathbb{K}}}) - 1.1714} & \text{if } \log(\Delta_{\mathbb{L}}^{1/n_{\mathbb{K}}}) > e^{1.1714}. \end{cases}$$

The last bound is a generalization of Robin's result

$$\omega(n) := \sum_{p|n} 1 \leq \frac{\log n}{\log\log n - 1.1714}.$$

By partial summation:

Theorem (Grenié-M.) *(GRH) For $x \geq 2$*

$$\Big|\frac{|G|}{|C|}\pi_C(x) - \int_2^x \frac{\mathrm{d}u}{\log u}\Big| \leq \sqrt{x}\Big(\Big(\frac{1}{2\pi} + \frac{3}{\log x}\Big)\log \Delta_{\mathbb{L}} + \Big(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x}\Big)n_{\mathbb{L}}\Big).$$

The bounds coming from the estimations for $\psi(C;x)$ prove that there is a prime ideal with a given Frobenius and a norm bounded by $(0.1 + o(1))(\log \Delta_{\mathbb{L}})^2(\log\log \Delta_{\mathbb{L}})^4$. Using $\psi^{(1)}(C;x)$ we can remove the double log term.

Theorem (Grenié-M.) *(GRH) Let $k \geq 1$. Then $\pi_C(x) \geq k$ when*

$$x \geq 1.16\Big(\log \Delta_{\mathbb{L}} + \Big(\frac{k+5}{3}\Big)|G| + 15\Big)^2.$$

Under the same hypotheses Bach proved that $\pi_C(x) \geq 1$ when

$$x \geq 1\big(\log \Delta_{\mathbb{L}} + \text{const.}\big)^2,$$

but its argument deals only $k = 1$.

Consequences

By partial summation:

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Thanks you for your attention