Arithmetic random waves and lattice points on spheres

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Nodal surface of toral Laplace eigenfunctions

$\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$: 3D flat torus.
$G : \mathbb{T}^3 \to \mathbb{R}$. **Nodal set**:

$$\{x \in \mathbb{T}^3 : G(x) = 0\}.$$

$\looparrowright$ stationary during membrane vibrations.

Study: nodal set of **Laplace eigenfunctions** G,

$$(\Delta + E)G = 0,$$

eigenvalue ('energy') $E > 0$, high energy limit $E \to \infty$.

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Examples of nodal surfaces

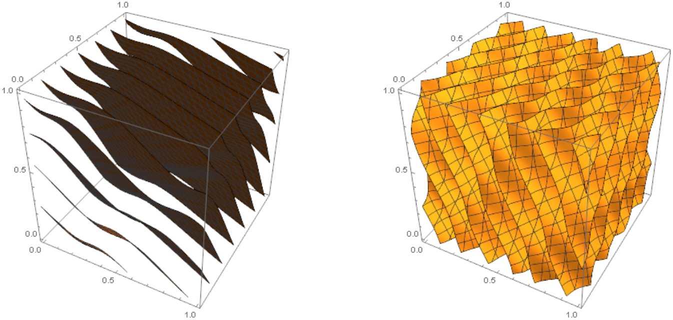

Figure: Nodal sets of
$G(x_1, x_2, x_3) = \cos[2\pi(2x_1 + 3x_2 + x_3)] + 2\sin[2\pi(x_1 + 3x_2 + 2x_3)]$;
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Energy $E = 4\pi^2(1^2 + 2^2 + 3^2)$.

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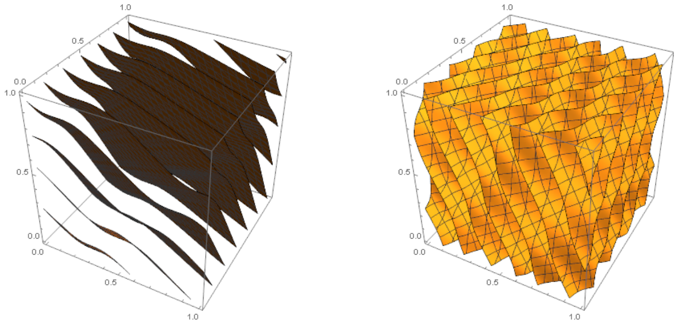

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Lattice points on spheres

Energy $E = 4\pi^2 m$. Set of **lattice points** on sphere $\sqrt{m}\mathcal{S}^2$:

$$\Lambda_m = \{\mu \in \mathbb{Z}^3 : \|\mu\|^2 = m\}; \qquad \mathcal{N}_m := |\Lambda_m| = r_3(m).$$

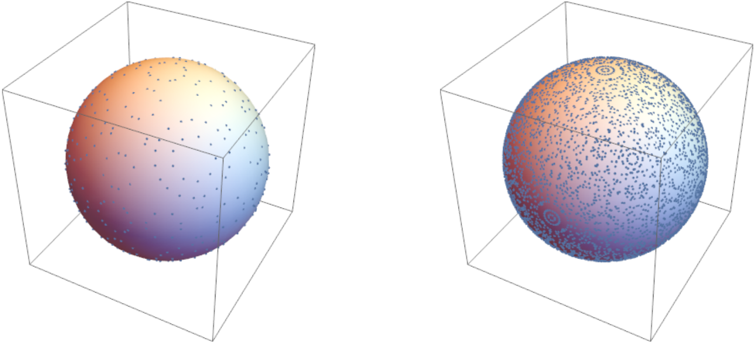

Figure: $m = 866$, $\mathcal{N} = 528$; $m = 146849$, $\mathcal{N} = 7392$.

The number of lattice points

$m = \square + \square + \square \iff m \neq 4^l(8k+7)$, and
$\mathcal{E}(4m) = \{2\mu : \mu \in \mathcal{E}(m)\} \looparrowright$ take $m \not\equiv 0, 4, 7 \pmod 8$.

$$(\sqrt{m})^{1-\epsilon} \ll \mathcal{N} \ll (\sqrt{m})^{1+\epsilon}, \qquad \forall \epsilon > 0.$$

Fix $R := \sqrt{m}$. Denote $\kappa(R)$ the maximal number of lattice points in intersection of sphere $R\mathcal{S}^2 \subset \mathbb{R}^3$ and plane Π:

$$\kappa(R) = \max_{\Pi} \left| \{\mu \in \mathbb{Z}^3 : \mu \in R\mathcal{S}^2 \cap \Pi\} \right|.$$

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Nodal intersections

Number of **nodal intersections** with $\mathcal{C}$ of fixed length,
$\mathcal{Z}(G) := |\{x \in \mathbb{T}^3 : G(x) = 0\} \cap \mathcal{C}|$, as $E \to +\infty$.

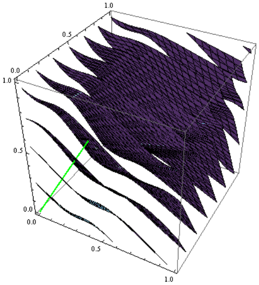

Figure: Nodal set of $G(x_1, x_2, x_3) = \cos[2\pi(2x_1 + 3x_2 + x_3)] + 2\sin[2\pi(x_1 + 3x_2 + 2x_3)]$ intersected with line of endpoints the origin and $(0.3, 0.2, \sqrt{3}/3)$.

There exist sequences of eigenfunctions G and curves $\mathcal{C}$, where $\mathcal{C} \subset$ nodal set, and planar curves with no nodal intersections at all, m arbitrarily large.

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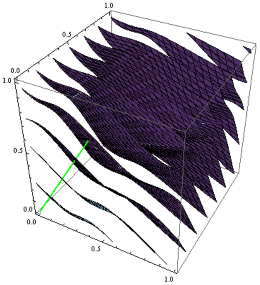

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Arithmetic random waves

Laplace eigenvalues $\{E = 4\pi^2 m\}$, m is **sum of 3 squares**.
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$\mathcal{N} = |\Lambda_m| = r_3(m)$.

Eigenspace: basis $\{e^{2\pi i \langle \mu, x \rangle}\}_{\mu \in \Lambda_m}$, dimension $\mathcal{N}$.

Eigenvalue multiplicities $\rightsquigarrow$ *random* Gaussian Laplace toral eigenfunctions ('arithmetic random waves'):

$$F_m(x) = \frac{1}{\sqrt{\mathcal{N}}} \sum_{\mu \in \Lambda_m} a_\mu e^{2\pi i \langle \mu, x \rangle}.$$

a_μ: i.i.d. complex std. Gaussian random variables ($a_{-\mu} = \overline{a_\mu}$).

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Rudnick-Wigman-Yesha

Statistics of nodal intersections $\mathcal{Z}(F) = |\{x \in \mathbb{T}^3 : F(x) = 0\} \cap \mathcal{C}|$.

- For smooth curves $\mathcal{C}$ of length L on the torus:

$$\mathbb{E}[\mathcal{Z}] = L\frac{2}{\sqrt{3}} \cdot \sqrt{m}.$$

- For $\mathcal{C}$ of nowhere-zero curvature, as $m \to \infty$, $m \not\equiv 0, 4, 7 \mod 8$,

$$\mathrm{Var}\left(\frac{\mathcal{Z}}{\sqrt{m}}\right) \ll \frac{1}{m^\delta}$$

where $\delta = 1/3$ if $\mathcal{C}$ of nowhere 0 torsion; any $\delta < 1/4$ if $\mathcal{C}$ is planar.
$\rightsquigarrow \mathcal{Z}/\sqrt{m}$ concentrates around its mean.

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Rational line segments

Nodal intersections number $\mathcal{Z}(F) = |\{x : F(x) = 0\} \cap \mathcal{C}|$ for arithmetic random waves F, against a straight line.

Theorem 1 (M.)

Assume the segment $\mathcal{C}$, of length L, is **rational** *i.e., it is parametrised by $\gamma(t) = t(\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_2/\alpha_1 \in \mathbb{Q}$ and $\alpha_3/\alpha_1 \in \mathbb{Q}$. Then*

$$Var\left(\frac{\mathcal{Z}}{\sqrt{m}}\right) \ll \frac{\kappa(\sqrt{m})}{\mathcal{N}}.$$

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Irrational line segments

Theorem 2 (M.)

Irrational line $\mathcal{C} \subset \mathbb{T}^3 : \gamma(t) = t(\alpha_1, \alpha_2, \alpha_3)$ *s.t.* $\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_1} \in \mathbb{R} \setminus \mathbb{Q}$. *If* $m \not\equiv 0, 4, 7 \pmod 8$, *then for every* $\epsilon > 0$,

$$Var\left(\frac{\mathcal{Z}}{\sqrt{m}}\right) \ll \frac{1}{m^{1/7-\epsilon}}.$$

Theorem 3 (M.)

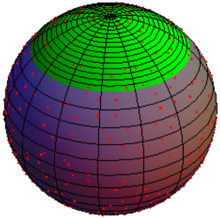
Irrational line $\mathcal{C} \subset \mathbb{T}^3 : \gamma(t) = t(\alpha_1, \alpha_2, \alpha_3)$ *s.t.* $\frac{\alpha_2}{\alpha_1} \in \mathbb{Q}$ *and* $\frac{\alpha_3}{\alpha_1} \in \mathbb{R} \setminus \mathbb{Q}$. *If* $m \not\equiv 0, 4, 7 \pmod 8$, *then for every* $\epsilon > 0$,

$$Var\left(\frac{\mathcal{Z}}{\sqrt{m}}\right) \ll \frac{1}{m^{1/5-\epsilon}}.$$

A conjecture of Bourgain and Rudnick

Conjecture 1: the maximal number of lattice points $\chi(R,s)$ in a radius s cap of $R\mathcal{S}^2$ satisfies, as $R\to\infty$, $\forall\epsilon>0$ and $s<R^{1-\delta}$:

$$\chi(R,s) \ll R^{\epsilon}\left(1+\frac{s^2}{R}\right).$$



↬ Proven for exponent $R^{1/2}$.

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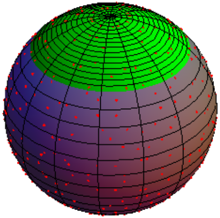

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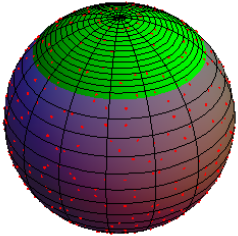
Conditional result

Theorem 4 (M.)

Assume Conjecture 1. Let $m \not\equiv 0, 4, 7 \pmod 8$ *and* $\mathcal{C}$ *be a straight line segment on* $\mathbb{T}^3$*. Then we have for all* $\epsilon > 0$

$$Var\left(\frac{\mathcal{Z}}{\sqrt{m}}\right) \ll \frac{1}{m^{1/4-\epsilon}}.$$

Spherical segments



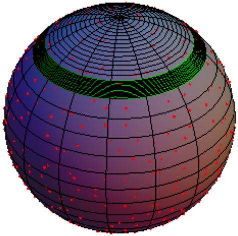


Figure: spherical cap; spherical segment.

Lattice points in spherical segments/1

Proposition 5 (M.)

In a sph. segment $S \subset R\mathcal{S}^2$ of angle θ, large base radius k, and direction β, s.t. $\frac{\beta_2}{\beta_1}, \frac{\beta_3}{\beta_1} \in \mathbb{R} \setminus \mathbb{Q}$, the lattice point number ψ satisfies

$$\psi \ll \kappa(R) \left(1 + R \cdot \theta^{1/3}\right)$$

for $\theta \to 0$.

Idea of proof: bound ψ for 'rational' sph. segments, then use Diophantine approximation.

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Simultaneous Diophantine approximation

Theorem 6 (Dirichlet)

Given $\zeta_1, \zeta_2 \in \mathbb{R} \setminus \mathbb{Q}$ *and an integer* $H \geq 1$*, there exist* $q, p_1, p_2 \in \mathbb{Z}$ *s.t.* $1 \leq q \leq H^2$ *and*

$$\left|\zeta_1 - \frac{p_1}{q}\right|, \ \left|\zeta_2 - \frac{p_2}{q}\right| < \frac{1}{qH}.$$

↬ If the line $\mathcal{C} : \gamma(t) = t\alpha$ satisfies $\alpha_2/\alpha_1 \in \mathbb{Q}$, we get a better bound, as we approximate one irrational only.

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Lattice points in spherical segments/2

Recall: $\chi(R, s) =$ max. number of lattice points in cap of radius s.

Proposition 7 (M.)

In a sph. segment $S \subset R\mathcal{S}^2$ of angle θ and large base radius k, the lattice point number ψ satisfies, for every real number $0 < \Omega < R$,

$$\psi \leq \chi(R, (2\pi + 1/2)\Omega) \cdot \left\lceil \frac{k}{\Omega} \right\rceil \cdot \left\lceil \frac{R\theta}{\Omega} \right\rceil$$

Idea of proof: cover the segment with spherical caps.

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Riccardo W Maffucci.
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