The G.C.D. of n and the nth Fibonacci number

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(joint work with Carlo Sanna)

Università di Milano "Luigi Bocconi"

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F_n and n

Let $(F_n)_{n \geq 1}$ be the sequence of Fibonacci numbers, defined as usual by

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In particular, the set of positive integers n such that $n \mid F_n$ has been studied by Alba González–Luca–Pomerance–Shparlinski, André-Jeannin, Luca–Tron, Somer.

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Lemma

$n \in \mathcal{A}$ if and only if $n = \gcd(\ell(n), F_{\ell(n)})$.

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Let us see a brief sketch of the proof...

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$$r(m) := \begin{cases} 1 & \text{if } 10 \nmid m, \\ 5/4 & \text{if } m \equiv 10 \bmod 20, \\ 1/2 & \text{if } 20 \mid m. \end{cases}$$

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Thanks to the previous Lemma, we have $\mathcal{P} \subseteq \mathcal{A} \cup \{3\}$. Hence, it is enough to prove that

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Therefore, by Mertens' theorem, we get that

$$\#\mathcal{P}_1(x) \gg \frac{1}{\log y} \cdot \frac{x}{\log x},$$

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By Cubre and Rouse's result, we have that $\mathcal{Q}$ has a positive relative density in the set of all primes. As a consequence, we can pick a sufficiently large $y > 0$ so that

$$\prod_{p \in \mathcal{Q}(y)} \left(1 - \frac{1}{p}\right) < \frac{\varepsilon}{2}.$$

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and, by the arbitraryness of ε, it follows that $\#\mathcal{A}(x) = o(x)$. $\square$

Reference and Open questions

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L. and Sanna, *On the greatest common divisor of n and the nth Fibonacci number*, Rocky Mountain J. Math. (accepted).

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