Finite Ramanujan expansions and shifted convolution sums

(joint work with M.Ram Murty & B. Saha)

Giovanni Coppola

University of Salerno

We define the *shifted convolution sum* (also, *correlation*) of any couple $f, g : \mathbf{N} \to \mathbf{C}$ as

$$C_{f,g}(N,a) \stackrel{def}{=} \sum_{n \le N} f(n)g(n+a).$$

The integer variable $a > 0$ is the *shift*.

There's a lack of asymptotic/explicit formulæ, for correlations of interesting f, g (esp., case $f = g = \Lambda$, the von-Mangoldt function, with even $a = 2k \ge 2$, involves $2k-$twin primes!), too difficult (apart special cases) to achieve, even for one single, fixed shift $a > 0$.

For $f' \stackrel{def}{=} f * \mu$ and $g' \stackrel{def}{=} g * \mu$ Möbius inversion

$$\Rightarrow \quad f(n) = \sum_{d|n} f'(d) \quad \text{and} \quad g(m) = \sum_{q|m} g'(q)$$

so : **vital remark** is that inside

$$(1) \quad C_{f,g}(N,a) = \sum_{d} f'(d) \sum_{q} g'(q) \sum_{\substack{n \le N \\ n \equiv 0 \bmod d \\ n \equiv -a \bmod q}} 1$$

$$= \sum_{d \le N} f'(d) \sum_{q \le N+a} g'(q) \sum_{\substack{n \le N \\ n \equiv 0 \bmod d \\ n \equiv -a \bmod q}} 1,$$

our $f(n)$, $g(m)$ become *truncated divisor sums*

$$\sum_{d|n, d \le N} f'(d), \qquad \sum_{q|m, q \le N+a} g'(q)$$

(depending on both variables, N and shift a); the condition $d|n$ can be expressed as

$$\mathbf{1}_{d|n} = \frac{1}{d}\sum_{j\le d} e_d(jn) = \frac{1}{d}\sum_{q|d} c_q(n),$$

involving *Ramanujan sums*

$$c_q(n) \stackrel{def}{=} \sum_{j\le q,(j,q)=1} e_q(jn),$$

after g.c.d. rearrangement, from *orthogonality* of *additive characters* $e_q(m) \stackrel{def}{=} e^{2\pi i m/q}$.

We immediately get **any arithmetic functions** f, g : **N** $\rightarrow$ **C** have (inside $C_{f,g}$) following *finite Ramanujan expansions* (exchanging sums now)

$$f(n) = \sum_{d\le N} f'(d)\mathbf{1}_{d|n} = \sum_{q\le N} \widehat{f}(q)c_q(n),$$

$$g(m) = \sum_{d\le N+a} g'(d)\mathbf{1}_{d|m} = \sum_{q\le N+a} \widehat{g}(q)c_q(m),$$

(finite expansions depending on N, a again)

with *Ramanujan coefficients*

$$\widehat{f}(q) \stackrel{def}{=} \sum_{\substack{d \le N \\ d \equiv 0 \bmod q}} \frac{f'(d)}{d}, \quad \widehat{g}(q) \stackrel{def}{=} \sum_{\substack{d \le N+a \\ d \equiv 0 \bmod q}} \frac{g'(d)}{d}.$$

Thus heuristic formula for f and g correlation

(2) $$C_{f,g}(N,a) \sim S_{f,g}(a)N,$$

with $a \ge 1$, defining the f and g *singular series*:

$$S_{f,g}(a) \stackrel{def}{=} \sum_{q=1}^{\infty} \widehat{f}(q)\widehat{g}(q)c_q(a).$$

This has been proved in our first work (with Murty & Saha, see JNT) for particular f, g.

Actually, it is the *singular sum* (after N, $\widehat{f}=0$)

$$S_{f,g}(a) = \sum_{q \le N} \widehat{f}(q)\widehat{g}(q)c_q(a).$$

On the other hand, it depends on N. But, this variable is implicit in f, g.

Aficionados of Hardy-Littlewood method will say: these are only partial sums of singular series!

Heuristic (2) inspired the definition:

$$(3)\quad C_{f,g}(N,a) = \sum_{\ell=1}^{\infty} \widehat{C_{f,g}}(N,\ell) c_\ell(a), \quad \forall a \in \mathbf{N}$$

which is the *shift-Ramanujan expansion* of our correlation. Notice: Hildebrand's Theorem ensures pointwise convergence! (For all arithmetic functions, here shift a is the argument)

(Big!) Problem is to find the *shift-Ramanujan coefficients* $\widehat{C_{f,g}}(N,\ell)$.

Now, we **don't know**, if (3) is a **finite** sum!

For this, *Carmichael formula*

$$\widehat{C_{f,g}}(N,\ell) = \frac{1}{\varphi(\ell)} \lim_{x\to\infty} \frac{1}{x} \sum_{m\le x} C_{f,g}(N,m) c_\ell(m)$$

is useful. Two pbs: 1) when? & 2) how?

Both questions need two **new concepts**: the *purity*, of a Ramanujan expansion, and the *fair correlations*.

We say a Ramanujan expansion is "pure", iff coefficients & their supports **do not depend** on outer variable. In other words, the variable we expand *appears only in Ramanujan sums*. In above (3) outer variable's the shift a.

Purity is a **strong** requirement: *finite & pure* Ramanujan exp.s *are truncated divisor sums*! (Hildebrand Th.m expands any $f(n)$ into finite Ramanujan exp. $\Rightarrow$ not pure: $n-$dependence)

Very similar is the definition: $C_{f,g}(N,a)$ is *fair* $\stackrel{def}{\Longleftrightarrow}$ $a-$dependence is only inside g argument $(n+a)$. Equivalently, $f(n)$ and $\widehat{g}(q)$ do not depend on a, neither in supports, in following

$$(4) \quad C_{f,g}(N,a) = \sum_q \widehat{g}(q) \sum_{n\le N} f(n) c_q(n+a).$$

This formula comes easily from the g finite Ramanujan expansion.

Our 2nd paper (C-Murty) proves the following.

Abbreviate Ramanujan expansion (3) as s.R.e.

Theorem 1. *Assume* $g(m) = \sum_{q|m,q\le Q} g'(q)$, Q *independent of* a *and* $C_{f,g}(N,a)$ *is fair. Then* F.A.E.

- s.R.e. *is pure & uniformly convergent;*

- s.R.e. *coefficients from Carmichael formula;*

- s.R.e. *has* **R***amanujan* **e***xact* **e***xplicit* **f***ormula:*

$$C_{f,g}(N,a) = \sum_{\ell\le Q} \frac{\widehat{g}(\ell)}{\varphi(\ell)} \sum_{n\le N} f(n)c_\ell(n)c_\ell(a) \forall a \in \mathbf{N}$$

- s.R.e. *is pure & finite.*

Definition: such a s.R.e. is *regular*.

Remark: Once found the **Reef**, we'd find the treasure (our's to prove (2) above) !

This is not a joke, but (for reasonable f, g) a consequence:

Corollary 1. *Same hypotheses of Theorem* 1 *give, for* $f(n) = \sum_{d|n,d\le D} f'(d)$, $\frac{\log D}{\log N} < 1-\delta$, *with regular s.R.e., whenever* f, g *satisfy the Ramanujan Conjecture,*

$$C_{f,g}(N,a) = S_{f,g}(a)N + O(N^{1-\delta}).$$

Notice "gain", $\delta > 0$, in remainder's exponent depends on f.

(In general they both work, for all g, taking $Q = N$ and cut as (1) on `arxiv`:1709.06445)

The case $f = g = \Lambda$ is not covered now (i.e., in II, Cor.1) but in `arxiv`:1709.06445 we are calculating $a-$primes ($a = 2k \geq 2$) correlation, from

$$\sum_{n \leq N} \Lambda(n) c_\ell(n) \approx \sum_{p \leq N} (\log p) c_\ell(p) \sim \mu(\ell) N$$

($\forall \ell$, apart "few cases", by PNT), in the **Reef** of $a-$twin primes. (Btw, $\widehat{\Lambda}(q) =$? See JNT)

Twin primes regularity gives H-L asymptotic! (On `arxiv`:1709.06445 we need Delange Hp, now even less: see following)

Since regularity is "hard", to prove, we come to "soft", say, hypotheses: work in progress.

Natural question: what can we prove only from "g is of range Q and fair correlation"? These (in Th.m 1), we call "basic hypotheses", give

$$C_{f,g}(N,a) = \sum_{\ell \le Q} \widehat{C_{f,g}}(N,Q,\ell) c_\ell(a) + \sum_{\substack{d|a \\ d>Q}} C'_{f,g}(d),$$

by Möbius inversion & $d|a$ formula, with **pure** "truncated Ramanujan coeff.s":

$$\widehat{C_{f,g}}(N,Q,\ell) \stackrel{def}{=} \sum_{\substack{d \le Q \\ d \equiv 0 \bmod \ell}} \frac{C'_{f,g}(d)}{d}$$

similar to **Wintner-Delange formula** (for infinite expansions, under hypotheses). Defined Eratosthenes transform of correlation as:

$$C'_{f,g}(d) = C'_{f,g}(N,d) \stackrel{def}{=} \sum_{t|d} C_{f,g}(N,t) \mu(d/t).$$

Then, (4) above, fairness and Carmichael for truncated $\widehat{C_{f,g}}$ (from purity) give $\forall q \in \mathbf{N}$

$$\widehat{C_{f,g}}(N,Q,q) = \frac{\widehat{g}(q)}{\varphi(q)} \sum_{n \le N} f(n) c_q(n) - L(q),$$

abbreviating $\forall q \in \mathbf{N}$

$$L(q) \stackrel{def}{=} \frac{1}{\varphi(q)} \lim_x \frac{1}{x} \sum_{m \le x} \sum_{d|m, d>Q} C'_{f,g}(d) c_q(m),$$

notice always exists $\in \mathbf{C}$ and vanishes (as $0-0$) on $q > Q$. **In all**, $C_{f,g}(N,a) =$

$$\sum_{q \le Q} \left(\frac{\widehat{g}(q)}{\varphi(q)} \sum_{n \le N} f(n) c_q(n) - L(q) \right) c_q(a) + \sum_{\substack{d|a \\ d>Q}} C'_{f,g}(d)$$

comes from "g of range Q and fair correlation".

Well, $L(q)$ is a kind of "grey box", since:

$$\frac{1}{x}\sum_{m\le x}\sum_{d|m,d>Q} C'_{f,g}(d)c_q(m)$$

$$= \sum_{Q<d\le x} C'_{f,g}(d)\cdot \frac{1}{x}\sum_{K\le \frac{x}{d}} c_q(dK)$$

$$= \sum_{Q<d\le x} C'_{f,g}(d)\left(\frac{\varphi(q)}{d}\mathbf{1}_{q|d} + O_q\left(\frac{1}{x}\right)\right),$$

from classical exponential sums cancellation. Then "Slow Decay", abbrev. SD,

$$\textbf{SD} \qquad \sum_{d\le x} |C'_{f,g}(d)| = o(x), \ \ x\to\infty$$

(tantamount to : $|C'_{f,g}(d)|$'s mean-value$=0$!)

is the right **further hypothesis** to get $L(q)$ and the new "beyond Ramanujan coeff.s":

$$L(q) = \sum_{\substack{d>Q \\ d\equiv 0 \bmod q}} \frac{C'_{f,g}(d)}{d} \stackrel{def}{=} \lim_{x\to\infty} \sum_{\substack{Q<d\le x \\ d\equiv 0 \bmod q}} \frac{C'_{f,g}(d)}{d}$$

This added to $\widehat{C_{f,g}}(N,Q,q)$ above gives the $q-$th "extended Ramanujan coeff.":

$$\sum_{d\equiv 0 \bmod q} \frac{C'_{f,g}(d)}{d} \stackrel{def}{=} \lim_{x\to\infty} \sum_{\substack{d\le x \\ d\equiv 0 \bmod q}} \frac{C'_{f,g}(d)}{d}$$

where $\exists \lim \in \mathbf{C}$ and $=0$ on $q>Q$, like beyond ones. **Extended are Wintner-Delange**!

In all, basic hypotheses & Slow Decay give explicitly Wintner-Delange coefficients

$$\sum_{d\equiv 0 \bmod \ell} \frac{C'_{f,g}(d)}{d} = \frac{\widehat{g}(\ell)}{\varphi(\ell)} \sum_{n\le N} f(n)c_\ell(n)$$

very easily from above! Short calculations **add**

$$C_{f,g}(N,a) = \sum_{\ell\le Q} \frac{\widehat{g}(\ell)}{\varphi(\ell)} \sum_{n\le N} f(n)c_\ell(n)c_\ell(a)$$

$$+\sum_{d>Q} \frac{C'_{f,g}(d)}{d} \sum_{\substack{\ell|d \\ \ell>Q}} c_\ell(a).$$

See, if we may exchange ℓ, d sums into

$$\sum_{d=1}^{\infty} \sum_{\ell|d} \frac{C'_{f,g}(d)}{d} c_\ell(a) = \sum_{\ell=1}^{\infty} \sum_{d\equiv 0 \bmod \ell} \frac{C'_{f,g}(d)}{d} c_\ell(a)$$

then on LHS detecting $\mathbf{1}_{\ell|d} = \frac{1}{d}\sum_{\ell|d} c_\ell(a)$ gives

$$\sum_{d=1}^{\infty} \frac{C'_{f,g}(d)}{d} \sum_{\ell|d} c_\ell(a) = \sum_{d|a} C'_{f,g}(d) = C_{f,g}(N,a),$$

with on RHS the Wintner-Delange coefficients

$$\sum_{d\equiv 0 \bmod \ell} \frac{C'_{f,g}(d)}{d} = \frac{\widehat{g}(\ell)}{\varphi(\ell)} \sum_{n\le N} f(n) c_\ell(n), \quad \forall \ell \in \mathbf{N}$$

thus giving, again, the **Reef**. Under which hypotheses ? For example, Wintner Assumption:

WA $$\sum_{d=1}^{\infty} \frac{|C'_{f,g}(d)|}{d} < \infty$$

because (key ingredient: vanishing after Q)

$$\sum_{\ell=1}^{\infty} \sum_{d\equiv 0 \bmod \ell} \frac{|C'_{f,g}(d)|}{d} \cdot |c_\ell(a)|$$

$$\le Q \max_{\ell\le Q} |c_\ell(a)| \sum_{d=1}^{\infty} \frac{|C'_{f,g}(d)|}{d} < \infty$$

$\Rightarrow$ double series abs.convergence, whence ℓ, d exchange. Notice, WA gives the explicit formula for coefficients, based on Carmichael formula; so, the hypotheses of Theorem 1 ensure that Carmichael formula reaches the Reef!

In all: "g of range Q, fair correlation & WA" $\Longrightarrow$ **The Reef**!

Actually, under basic hypotheses, WA is the fifth condition to express regularity !

Advantage of WA on other conditions is, of course: "easy check".

(One technical, last comment's : $WA \Longrightarrow SD$)

THANKS!!!

[1] Coppola, G., Murty, M.Ram and Saha, B. - *Finite Ramanujan expansions and shifted convolution sums of arithmetical functions* - JNT

[2] Coppola, G. and Murty, M.Ram - *Finite Ramanujan expansions and shifted convolution sums of arithmetical functions, II* - to appear on JNT (see `arxiv` meanwhile)

[3] Coppola, G. - *An elementary property of correlations* - `arxiv`:1709.06445