On the non-vanishing of certain Dirichlet series

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Non-vanishing of Dirichlet series

Given a function $f : \mathbb{Z} \to \mathbb{C}$, $f(n) \ll n^{\varepsilon}$, we define

$$D(f,s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \qquad \Re(s) > 1.$$

In many cases D can be extended analytically past its domain of absolute convergence. Often in these cases it's important to know whether $D(f,1) = 0$.

The most important examples are the cases $f(n) = \mu(n)n^{it}$ and $f(n) = \chi(n)$, for which is well known that

$$D(\,(\cdot)^{it}\mu, 1) \neq 0 \qquad \text{(prime number theorem)}$$

$$D(\chi, 1) \neq 0 \qquad \text{(infinitude of primes in arithmetic progressions)}$$

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Chowla's problem

Chowla asked then the following question:

Problem (Chowla)

Let p be prime and let $f : \mathbb{Z} \to \mathbb{Q}$ periodic mod p with $f \not\equiv 0$. Then $D(f, 1) \neq 0$.

Notice that if $\sum_{a=1}^{p} f(a) \neq 0$, then $D(f, s)$ has a pole in $s = 1$.

In 1959 Chowla proved this in the case when

1. f odd
2. $\frac{p-1}{2}$ prime
3. $f : \mathbb{Z} \to \{\pm 1\}$

He sent this to Siegel who replied giving an argument which allowed to remove the conditions 2) and 3), i.e. solving Chowla's problem in the case of f odd. Chowla published Siegel's argument in 1964 and gave a new simplified argument in 1970.

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Cholwa's problem

Theorem (Chowla - Siegel)

Let p be prime and let $f : \mathbb{Z} \to \mathbb{Q}$ odd and periodic mod p with $f \not\equiv 0$. Then $D(f,1) \neq 0$.

The difference of the cases when f is odd and when f is even (or neither) can be well understood by the decomposing f into Dirichlet characters. If $f(0) = 0, \sum_{a=1}^{p} f(a) = 0$ we have

$$f(n) = \sum_{\chi_0 \neq \chi \mod p} c_\chi \chi(n) \qquad \forall n \in \mathbb{Z}$$

with $c_\chi \in \mathbb{Q}(\xi_{p-1})$ and $\xi_m = e^{2\pi i/m}$. Also, one has that f is odd iff $c_\chi = 0$ for all χ even and f is even iff $c_\chi = 0$ for all χ odd. Also,

$$D(f,1) = \sum_{\chi \mod p} c_\chi L(1,\chi)$$

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Chowla's problem

Now, if χ is odd and primitive we have

$$L(1,\chi) = \frac{\pi i \tau(\chi)}{p} \sum_{a=1}^{p} \chi(a) \left\{ \frac{a}{p} \right\}$$

and thus $L(1,\chi) \in \pi \cdot \mathbb{Q}(\xi_p, \xi_{p-1})$. If χ is even, we have

$$L(1,\chi) = A_\chi \sum_{1<a<p/2} \overline{\chi}(a) \log \eta_a$$

where $A_\chi \in \overline{\mathbb{Q}}$ for $\{\eta_a \mid 1 < a < p/2\}$ is a set of real multiplicatively independent units in the cyclotomic field (Ramachandra).

In particular, the proof of the odd case involves some algebraic number theory, whereas the proof of the even case is related to transcendence number theory and in particular to linear forms in logarithms.

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The general case

Theorem (Baker, Birch, Wirsing - 1973)

Let p be prime and let $f : \mathbb{Z} \to \mathbb{Q}$ be periodic mod p with $f \not\equiv 0$. Then $D(f,1) \neq 0$.

Proof (variation from Murty-Murty '11). We can assume f is not odd. By the above, we have that

$$D(1,f) = \pi r + \sum_{\substack{\chi_0 \neq \chi \bmod p \\ \chi \text{ even}}} c_\chi L(1,\chi)$$

$$= \pi r + \sum_{1<a<p/2} \log \eta_a \sum_{\substack{\chi_0 \neq \chi \bmod p \\ \chi \text{ even}}} c_\chi A_\chi \overline{\chi}(a)$$

with $r \in \overline{\mathbb{Q}}$. By Baker's theorem the values $\log \eta_a$ are linearly independent over $\overline{\mathbb{Q}}$ and are also independent from $\pi = \frac{1}{i}\log(-1)$. It follows that $L = 0$ if and only if the inner sum is zero for all χ and this is equivalent to if $c_\chi A_\chi = 0$ for all χ which can't happen because f is not odd.

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Consequences

The proof works also if $f : \mathbb{Z} \to K$ where K is a number field with $K \cap \mathbb{Q}(\xi_p) = \mathbb{Q}$ and f is periodic modulo q (not necessary prime) with $f(n) = 0$ whenever $1 < (n, q) < q$.

Corollary

Let p be prime. Then, the numbers $L(1, \chi)$, as χ varies among primitive characters mod p, are linearly independent over $\mathbb{Q}$.

- Okada '82 and Chatterjee and Murty '12: criteria for the case of non-prime periods.
- Murty, Saradha '07: transcendental values of the digamma function.
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$$D_k(f,s) := \sum_{n=1}^{\infty} \frac{d_k(n)f(n)}{n^s} \qquad \Re(s) > 1,$$

where $d_k(n)$ is the k-th divisor function.

Remark

If f is periodic mod q and $k > 1$ then $D_k(f,s)$ is holomorphic at $s = 1$ if and only if $\sum_{a=1}^{q} f(a) = 0$ and $f(0) = 0$.

The case where f is not odd can be tackled in exactly the same way, but we need a replacement for Baker's theorem.

Conjecture (Schanuel's conjecture)

Let $z_1, \ldots, z_n$ be linearly independent over $\mathbb{Q}$, then $\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$ has transcendence degree n.

In particular, if $\eta_1, \ldots, \eta_r \in \overline{\mathbb{Q}}$ are linearly independent then $\log \eta_1, \ldots, \log \eta_r$ are algebraically independent.

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A generalization

Proposition

Assume Schanuel's conjecture. Let p be prime and let $f : \mathbb{Z} \to \mathbb{Q}$ be periodic mod p, f **not odd**. *Then $D_k(f, 1) \neq 0$.*

Thus, the only case open (at least conditionally) is the case of f odd.

$$\sum_{\substack{n \in \mathbb{Z} \\ n \equiv 1 \mod 5}} \frac{d(|n|)}{n} = 2 \sum_{\substack{n \in \mathbb{Z} \\ n \equiv 2 \mod 5}} \frac{d(|n|)}{n} = \frac{4\pi^2}{25\sqrt{5}}$$

In particular, if f is the 5-periodic function with

$$f(1) = -f(-1) = 1, \qquad f(2) = -f(-2) = -2$$

then $D_2(f, 1) = 0$. Similarly, if f is the 13-periodic function with

$$f(1) = 18a, \qquad f(4) = 18b, \qquad f(3) = 18c$$
$$f(2) = 19a + 11b + 4c, \quad f(8) = -4a + 19b + 11c, \quad f(6) = -11a - 4b + 19c$$

for any $a, b, c \in \mathbb{C}$, then $D_2(f, 1) = 0$.

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Theorem (B., Martin '17)

Let p be prime and $V = \{f : \mathbb{Z} \to \mathbb{Q} \mid f$ odd and periodic mod $p\}$. Let $V_0 := \{f \in V \mid L_k(1, f) = 0\}$. Then,

$$\dim_{\mathbb{Q}}(V_0) \geq \begin{cases} \dim_{\mathbb{Q}}(V)\frac{r-1}{r} & \text{if } v_2(p-1) > v_2(k), \\ \dim_{\mathbb{Q}}(V)\frac{r-2}{r} & \text{if } v_2(p-1) \leq v_2(k), \end{cases}$$

where $r = (k, p-1)$ and $v_2(a)$ denotes the 2-adic valuation of a.

Moreover, the equality holds if $(k, p-1) \leq 2$ or if $(k, p-1) = 4$ and $p \equiv 5 \bmod 8$. In particular, $\dim_K(V_0) = 0$ if and only if $(k, p-1) = 1$ or if $(k, p-1) = 2$ and $p \equiv 3 \bmod 4$.

Corollary

Let p be prime with either $(k, p-1) \leq 2$ or $p \equiv 5 \bmod 8$ and $(k, p-1) = 4$. Then the set of values $L(1, \chi)^k$ are linearly independent over $\mathbb{Q}$ for χ that runs through the odd Dirichlet characters mod p.

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Sketch of the proof

Let $$x_k(r;p) := \frac{1}{p^k} \sideset{}{^*}\sum_{\substack{m_1,\dots,m_k \bmod p \\ m_1\cdots m_k \equiv r \bmod p}} \cot\left(\pi \frac{m_1}{p}\right)\cdots\cot\left(\pi\frac{m_k}{p}\right).$$

In particular, $x_1(r;p) = \frac{1}{p}\cot(\frac{\pi r}{p})$ and

$$x_2(r;p) := \frac{1}{p^2} \sideset{}{^*}\sum_{m \bmod p} \cot\left(\pi\frac{m}{p}\right)\cot\left(\pi\frac{r\overline{m}}{p}\right).$$

Proposition

Let $k \in \mathbb{N}$, p be a prime and $r \in \mathbb{Z}$ with $(r,p) = 1$. Then

$$x_k(r;p) = \frac{1}{2}\left(\frac{2}{\pi}\right)^k \sum_{\substack{n\in\mathbb{Z} \\ n\equiv r \bmod p}} \frac{d_k(|n|)}{n}.$$

In particular, if $f : \mathbb{Z} \to \mathbb{C}$ is odd and periodic modulo p, then

$$D_k(1,f) = 2\left(\frac{\pi}{2}\right)^k \sum_{r=1}^{(p-1)/2} f(r)x_k(r;p).$$

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$$D_k(1,f) = 2\left(\frac{\pi}{2}\right)^k \sum_{r=1}^{(p-1)/2} f(r)x_k(r;p).$$

Sketch of the proof

Note that $i^k x_k(r;p) \in \mathbb{Q}(\xi_p)$. Moreover, given $c \in \mathbb{Z}$, the action of the Galois automorphism $\sigma_c : \mathbb{Q}(\xi_p) \to \mathbb{Q}(\xi_p)$ with $\xi_p \mapsto \xi_p^c$ satisfies

$$\sigma_c(i^k x_k(r;p)) = i^k x_k(c^k r;p).$$

In particular, if $f : \mathbb{Z} \to \mathbb{Q}$ is odd with $D(1,f) = 0$ then

$$\sum_{r=1}^{(p-1)/2} f(r) x_k(r;p) = 0$$

Or equivalently, assuming g is a generator of $(\mathbb{Z}/p\mathbb{Z})^*$,

$$\sum_{j=0}^{\frac{p-3}{2}} f(g^j) x_k(g^{j+k\ell};p) = 0 \qquad 0 \le \ell < \frac{p-1}{(p-1,k)}$$

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Thus, we have a system of $\frac{p-1}{(p-1,k)}$ or $\frac{p-1}{2(p-1,k)}$ equations in $\frac{p-3}{2}$ variables.

$$\sum_{j=0}^{\frac{p-3}{2}} f(g^j) x_k(g^{j+k\ell}; p) = 0 \qquad 0 \le \ell < \frac{p-1}{(p-1,k)}$$

Moreover, this system is invariant when applying any automorphism of $\mathbb{Q}(\xi_p)$. In particular, the space of zeros over $\mathbb{Q}$ has dimension $\ge \frac{p-1}{2} - \frac{p-1}{(p-1,k)}$ or $\ge \frac{p-1}{2} - \frac{p-1}{2(p-1,k)}$.

To give a specific example we consider the case $k = 2$. We have two possibilities: if $p \equiv 3 \bmod 4$ (i.e. $v_2(p-1) \le v_2(k)$) then we have a system of $\frac{p-1}{2}$ equations in $\frac{p-1}{2}$ variables. If $p \equiv 1 \bmod 4$ (i.e. $v_2(p-1) > v_2(k)$), then for $0 \le \ell < \frac{p-1}{4}$

$$\sum_{j=0}^{\frac{p-3}{4}} f(g^{2j}) x_k(g^{2(\ell+j)}; p) + \sum_{j=0}^{\frac{p-3}{4}} f(g^{1+2j}) x_k(g^{1+2(\ell+j)}; p) = 0$$

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This translate into

$$(x_k(g^{2(i+j)};p))_{i,j}\begin{pmatrix} f(g^2) \\ \vdots \\ f(g^{\frac{p-3}{2}}) \end{pmatrix} + (x_k(g^{1+2(i+j)};p))_{i,j}\begin{pmatrix} f(g) \\ \vdots \\ f(g^{1+\frac{p-3}{2}}) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and thus we have to understand

$$\det((x_k(g^{a+2(i+j)};p))_{i,j\le\frac{p-3}{4}}) \quad a=0,1 \qquad \text{for } p\equiv 1 \mod 4$$

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If we can prove that these determinants are non-zero, then one obtains the equality in the Theorem when $k=2$.

Notice that since $g^{2\cdot p-12}=1$ and $g^{2\cdot\frac{p-1}{4}}=-1$, then these matrices are variations of circulant matrices.

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Lemma

For $m \geq 1$ let

$$A_+(\boldsymbol{v}) := \begin{pmatrix} v_0 & v_1 & \cdots & v_{m-1} \\ v_1 & v_2 & \cdots & v_0 \\ \vdots & \vdots & & \vdots \\ v_{m-1} & v_0 & \cdots & v_{m-2} \end{pmatrix} \quad A_-(\boldsymbol{v}) := \begin{pmatrix} v_0 & v_1 & \cdots & v_{m-2} & v_{m-1} \\ v_1 & v_2 & \cdots & v_{m-1} & -v_0 \\ \vdots & \vdots & & \vdots & \vdots \\ v_{m-1} & -v_0 & \cdots & -v_{m-3} & -v_{m-2} \end{pmatrix}.$$

Then,

$$\det(A_+(\boldsymbol{v})) = (\sin(\tfrac{\pi m}{2}) - \cos(\tfrac{\pi m}{2})) \prod_{\ell=0}^{m-1} \left(\sum_{j=0}^{m-1} v_j \xi_m^{j\ell} \right),$$

$$\det(A_-(\boldsymbol{v})) = (\sin(\tfrac{\pi m}{2}) + \cos(\tfrac{\pi m}{2})) \prod_{\substack{\ell=0 \\ \ell \text{ odd}}}^{2m} \left(\sum_{j=0}^{m-1} v_j \xi_{2m}^{j\ell} \right).$$

Circulant matrices

Applying this to our matrices we find

$$\det((x_k(g^{a+2(i+j)};p))_{i,j\leq\frac{p-3}{4}}) = (\sin(\tfrac{\pi(p-1)}{8}) + \cos(\tfrac{\pi(p-1)}{8}))\frac{2^{(k-2)(p-1)/4}}{\pi^{k(p-1)/4}} \times \prod_{\substack{\ell=0\\ \ell \text{ odd}}}^{(p-1)/2} \left(L(1,\chi_*^{\ell})^2 + (-1)^a L(1,(\tfrac{\cdot}{p})\chi_*^{\ell})^2\right).$$

where χ_* is a generator of the group of characters mod p.

Factoring and using that $L(1,\chi_*^{\ell})$ and $L(1,(\frac{\cdot}{p})$ are linearly independent over $\mathbb{Q}(i)$ it follows that the determinant is non-zero.

$$\det((x_k(g^{2(i+j)};p))_{i,j\leq\frac{p-3}{2}}) = (-1)^{(p-3)/4}\left(\tfrac{2}{\pi^2}\right)^{\frac{p-1}{2}} \prod_{\substack{\ell=0\\ \ell \text{ odd}}}^{p-1} L(1,\chi_*^{\ell})^2$$

$$= (-1)^{(p-3)/4} p^{-\frac{p+3}{2}} 2^{\frac{3p-7}{2}} (h_p^-)^2$$

where h_p^- is the relative class number.

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Open problems

Recall that

$$x_2(r;p) = \frac{2}{\pi^2} \sum_{\substack{n\in\mathbb{Z} \\ n\equiv r \mod p}} \frac{d(|n|)}{n} = \frac{1}{p^2} \sum_{m_1 \mod p}^{*} \cot\left(\pi \frac{m}{p}\right) \cot\left(\pi \frac{r\overline{m}}{p}\right).$$

Problem

How are the values in the set $\{x_2(r;p) \mid 1 \le r < p\}$ distributed as $p \to \infty$?

Note that it's fairly easy to compute all the moments of $x_2(r;p)$, but they don't determine the distribution of $x_2(r;p)$.

$$\sum_{r \mod p} x_2(r;p)^m = \left(\frac{2}{\pi^2}\right)^m \sum_{n\ge 1} \frac{d(n)^m}{n^m} + O_{m,\varepsilon}(p^{-1+\varepsilon}),$$

Open problems

Problem (Erdös)

Let $f : \mathbb{Q} \to \{\pm 1\}$ be periodic mod q. Is true that $D_1(1, f) \neq 0$?

This is (a subcase of) Chowla's problem when q is prime and it is known unless $q \equiv 1 \mod 4$. Okada proved that if q is square-free $D_1(1, f) = 0$ if and only if

$$\sum_{n | q^\infty} \frac{f(an)}{n} = 0 \qquad \forall a \text{ s.t. } (a, q) = 1$$

$$\sum_{\substack{r=1 \\ (r,q)=1}}^{q} f(r) = 0$$

Murty and Chatterjee proved that Erdös problem is verified for $> 82\%$ of $q \equiv 1 \mod 4$.

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Problem

What can we say about the function $f(X)$ defined below?

$$f(X) := \min\left(\left\{\left|\sum_{n\leq X}\frac{\epsilon_n}{n}\right| \;\middle|\; \epsilon_n \in \{\pm 1\}\right\}\right).$$

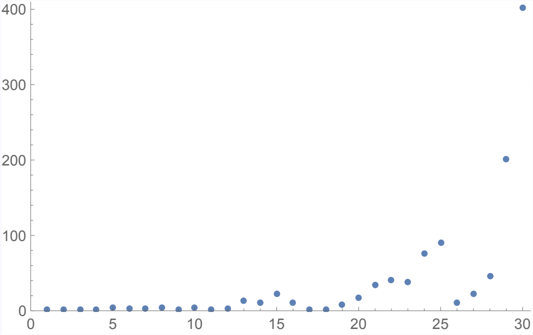

Figure: Graph of $f(n) * 2^n$ for $1 \leq n \leq 30$

Thanks!