Low discriminants for number fields of degree 8 and signature $(2,3)$

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Introduction

Let K be a number field of degree n and signature (r_1, r_2), where

- $r_1 := \#$ real embeddings of K.
- $r_2 := \#$ couples of complex conjugated embeddings of K.
- $n = r_1 + 2r_2$.

Theorem (Minkowski)

We have the inequality

$$\sqrt{|d_K|} \geq \frac{n^n}{n!}\left(\frac{\pi}{4}\right)^{r_2} =: M(n, r_2) \qquad \textit{(Minkowski's bound)}$$

where $M(n, r_2) > 1$ *for* $n \geq 2$ *and* $0 \leq r_2 \leq (n - r_1)/2$.

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Corollary

For every number field K *of degree* ≥ 2 *there is a prime number* $p \in \mathbb{Z}$ *which ramifies in* $\mathcal{O}_K$.

The problem of minimum discriminant

- **What is the minimum value of $|d_K|$ for a number field K of degree n?** Surely $|d_K| \geq M(n, r_2)^2$.
- If n is fixed and r_1 increases, then also $M(n, r_2) = \frac{n^n}{n!} \left(\frac{\pi}{4}\right)^{r_2}$ increases.
- **What is the minimum value of $|d_K|$ for a number field K of degree n and with r_1 real embeddings?**

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- **What is the minimum value of $|d_K|$ for a number field K of degree n and with r_1 real embeddings?**
- The problem has been solved for $n \leq 7$, with any signature, and for $n = 8$, with signature $(8, 0)$ or $(0, 4)$.
- The minimal case which is not completely known is $n = 8$ and $(r_1, r_2) = (2, 3)$.

Hunter-Pohst-Martinet method

Let K be a number field of degree n, and $\alpha \in \mathcal{O}_K \setminus \mathbb{Z}$. Let $f(x) := x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ be its minimum polynomial.
Is it possible to bound the coefficients of $f(x)$ through the discriminant of K?

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Is it possible to bound the coefficients of $f(x)$ through the discriminant of K?

- $a_n = \mathrm{N}(\alpha)$.
- **Symmetric functions**: for every $m \in \mathbb{Z}$ define

$$S_m(\alpha) := \sum_{i=1}^{n} \alpha_i^m.$$

(where $\alpha_i := \sigma_i(\alpha)$). We have the congruence relations

- $a_1 = -S_1(\alpha) = -\mathrm{Tr}(\alpha)$
- $S_m = -ma_m - \sum_{i=1}^{m-1} a_{m-i}S_i \qquad$ for $2 \leq m \leq n$.
- $S_m = -\sum_{i=1}^{n} a_i S_{m-i} \qquad$ for $m > n$

The goal is to bound the symmetric functions.

Hunter-Pohst-Martinet method

Define $T_m(\alpha) := \sum_{i=1}^{n} |\alpha_i|^m$ for every $m \in \mathbb{Z}$ (**absolute symmetric functions**). Obviously $|S_m(\alpha)| \leq T_m(\alpha)$.
The function T_m goes from $\mathcal{O}_K$ to $\mathbb{R}$, and T_2 is a quadratic form on the lattice induced in $\mathbb{R}^{r_1+r_2}$ by the embeddings.

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Theorem (Hunter-Pohst, 1982)

Let K be a number field of degree n and discriminant d_K. Then there exists $\alpha \in \mathcal{O}_K \setminus \mathbb{Z}$ such that

$$0 \leq \mathrm{Tr}(\alpha) \leq \frac{n}{2},$$

$$T_2(\alpha) \leq \frac{(\mathrm{Tr}(\alpha))^2}{n} + \gamma_{n-1} \left| \frac{d_k}{n} \right|^{1/(n-1)} =: U_2$$

where γ_{n-1} is the $(n-1)$-th Hermite's constant.

Remark: Martinet gave a stronger result when K has proper subfields.

Hunter-Pohst-Martinet method

Theorem

Let $T, N > 0$ *be such that* $N \leq (T/n)^{n/2}$*. Then,* $\forall m \in \mathbb{Z} \setminus \{0, 2\}$*, the function* $T_m(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i^m$ *has a global maximum over*

$$S := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon \sum_{i=1}^{n} x_i^2 \leq T, \prod_{i=1}^{n} x_i = N, x_i \geq 0 \text{ for every } i\}$$

and this maximum is attained in a point $(y_1, \ldots, y_n) \in S$ *with at most two different values for the coordinates.*

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Assume $T_2(\alpha) \leq T$. For every integer $1 \leq t \leq n-1$ we look for the least positive root of

$$t(y^{t-n}N)^{2/t} + (n-t)y^2 - T = 0$$

and we call it $y_1(t)$. Then, $\forall m \in \mathbb{Z} \setminus \{0,2\}$ one has

$$T_m(\alpha) \leq U_m := \max_{1 \leq t \leq n-1} [t(y_1(t)^{t-n}N)^{m/t} + (n-t)y_1(t)^m].$$

Stark-Odlyzko-Poitou-Serre's method

For every number field K we define the **Dedekind Zeta function**

$$\zeta_K(s) := \sum_{I \subset \mathcal{O}_K} \frac{1}{\mathrm{N}(I)^s} = \prod_{\mathcal{P} \subset \mathcal{O}_K} \left(1 - \mathrm{N}(\mathcal{P})^{-s}\right)^{-1}$$

where $\mathrm{N}(I) := \#\mathcal{O}_K / I$, $s \in \mathbb{C}$ and $\mathcal{P}$ ranges are the prime ideals in $\mathcal{O}_K$.
Let $f : \mathbb{R} \to \mathbb{R}$ be positive, even, $f(0) = 1$, with suitable growth and mean conditions and with positive Fourier transform.

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Lemma

For K of degree n and signature (r_1, r_2), for every $y > 0$, we have:

$$\begin{aligned}
\frac{1}{n}\log|d_K| \geq \gamma + \log(4\pi) + \frac{r_1}{n} \\
&- \int_0^\infty (1 - f(x\sqrt{y}))\left(\frac{1}{\sinh(x)} + \frac{r_1}{n}\frac{1}{2\cosh^2(x/2)}\right)dx \\
&- \frac{4}{n}\int_0^\infty f(x\sqrt{y})dx + \frac{4}{n}\sum_{\mathcal{P},m} \frac{\log(\mathrm{N}(\mathcal{P}))}{1 + (\mathrm{N}(\mathcal{P})^m)} f(m \log \mathrm{N}(\mathcal{P})\sqrt{y}).
\end{aligned}$$

Local corrections

- **Best known choice for** f (Tartar, 1973):

$$f(x) := \left(\frac{3}{x^3} (\sin(x) - x\cos(x)) \right)^2$$

  the square of the Fourier transform of $u(x) := (1 - x^2)\chi_{|x| \leq 1}(x)$.
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- The presence of a prime ideal $\mathcal{P}$ gives a **local correction** to the lower bound.
- Selmane (1999) used this inequality to compute the following lower bounds for $|d_K|$, whenever K has $n = 8$, $(r_1, r_2) = (2, 3)$ and admits a prime ideal $\mathcal{P}$ of norm $\mathrm{N}(\mathcal{P})$:

$\mathrm{N}(\mathcal{P})$	$\|d_K\| >$
2	11725962
3	8336752
4	6688609
5	5726300
7	4682934.

The main goal

We want to detect every number field with $n = 8$, signature $(2, 3)$ and $|d_K| \leq 5726300$. The idea is to range all the possible values for the symmetric functions S_m in the intervals $[-U_m, U_m]$, and use them to create the polynomials $p(x)$, which subsequently must be examined. There are some preliminary issues to underline:

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- The polynomials must be monic and we set an integer value between 0 and -4 for a_1 (remember that $a_1 = -\operatorname{Tr}(\alpha)$).
- We set $T := U_2$ and $N := |a_8| = |\operatorname{N}(\alpha)|$ such that $N \leq (U_2/8)^4$ (arithmetical-geometrical means inequality).
  By Selmane's estimates, N cannot be an exact multiple of $2, 3, 4$ or 5. One verifies that $N = 1$ (unless $a_1 = -3, -4$, in this case also $N = 7, 8, 9$ are admissible).

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- The procedure may miss the minimum polynomial of a field with proper subfields; but these fields are already classified by Algorithmic Class Field Theory and Martinet's Theorem (in fact, we detect them anyway).

Algorithmic steps

From now on, we assume $N = 1$, and that all the polynomials evaluated in 1 return an odd number.

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0) Given S_1, we have $a_1 = -S_1$.
We set the value for U_2 and then compute the bounds U_m for the absolute symmetric functions. We have then the intervals $[-U_m, U_m]$ (with $m \in \{2, \ldots, 8\}$ and $m \in \{-1, -2\}$). Select $a_8 \in \{-1, 1\}$.

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1) Let S_2 be the maximum positive integer in $[-U_2, U_2]$ such that $S_2 = -a_1 S_1 \mod 2$. Then put $a2 = (-S_2 - a_1 S_1)/2$.
   Call S_3 the maximum positive integer in $[-U_3, U_3]$ such that $S_3 = -a_1 S_2 - a_2 S_1 \mod 3$. Then put $a_3 := (-S_3 - a_1 S_2 - a_2 S_1)/3$.
   Do the same for S_4 up to S_7, creating a_4 up to a_7. Let $p(x) := x^8 + a_1 x^7 + a_2 x^6 + a_3 x^5 + a_4 x^4 + a_5 x^3 + a_6 x^2 + a_7 x + a_8$ be the polynomial to be checked.

Remark: If $p(1)$ is even, discard this polynomial and create the next by increasing a_7 of 1 (and so decreasing S_7 of 7).

Algorithmic steps

2) Before saying how to check $p(x)$, let us show how to go on the next polynomial.

- To create the next polynomial, one just has to increase a_7 of 2, decreasing then S_7 of 14, and keeping the previous coefficients.
- Check and repeat this way until $S_7 < -U_7$: then increase a_6 of 1 and decrease S_6 of 6, and compute a new S_7 and a new a_7, for which you can repeat what explained before.
- Do so until $S_6 < -U_6$: then increase a_5 of 1, decreasing S_5 of 5, and compute new S_6, a_6, S_7 and a_7. Then repeat the previous steps.
- And so on for every S_m, until $S_m < -U_m$ (with $m \geq 2$).

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- Do so until $S_6 < -U_6$: then increase a_5 of 1, decreasing S_5 of 5, and compute new S_6, a_6, S_7 and a_7. Then repeat the previous steps.
- And so on for every S_m, until $S_m < -U_m$ (with $m \geq 2$).

During the construction of the S_m's one can already check the following:

- If $a_1 = 0$, then $S_3 \geq 0$.
- $S_2 \geq -U_2 + \frac{2}{n} a_1^2$
- $|S_3| \leq \left(\frac{S_2+T}{2}(S_4 + 2(T - S_2)^2)\right)^{1/2}$.
- $S_4 \geq -2(T - S_2)^2$.

Algorithmic steps

3) If $p(x)$ misses one of the following conditions, then it has to be discarded.

- $|p(1)| = |\operatorname{N}(\alpha - 1)| \leq ((U_2 - 2S_1 + 8)/8)^4$ and it must be an admissible norm for a field with $|d_K| \leq 5726300$.
- $|p(-1)| = |\operatorname{N}(\alpha + 1)| \leq ((U_2 + 2S_1 + 8)/8)^4$ and it must be an admissible norm for a field with $|d_K| \leq 5726300$.

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- $|p(-1)| = |\,\mathrm{N}(\alpha + 1)| \leq ((U_2 + 2S_1 + 8)/8)^4$ and it must be an admissible norm for a field with $|d_K| \leq 5726300$.
- $-a_7/a_8 = S_{-1} \in [-U_{-1}, U_{-1}]$ and $(a_7^2/a_8 - 2a_6)/a_8 = S_{-2} \in [-U_{-2}, U_{-2}]$.
- $p(2), p(-2), p(3), p(-3), p(4), p(-4), p(5), p(-5)$ must be admissible norms.
- $-8a_8 - S_1a_7 - S_2a_6 - S_3a_5 - S_4a_4 - S_5a_3 - S_6a_2 - S_7a_1 = S_8 \in [-U_8, U_8]$.

If $p(x)$ satisfies every condition, then it is saved.

Algorithmic steps

Remark: Further conditions could be set, but it was not done in order to guarantee a reasonable time of computation (the worst case scenario, when $S_1 = 4$, takes less than two hours).
All these computations were done in MATLAB.

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All these computations were done in MATLAB.

4) The .mat files are then translated into .gp files and read by PARI/GP. For every polynomial $p(x)$ left, one finally checks if:
    - $p(x)$ is irreducible.
    - The discriminant d_K of the number field generated by $p(x)$ is negative (remember that $r_2 = 3$).
    - $d_K \geq -5726300$.
5) The few polynomials remaining define number fields which are classified via their isomorphism classes (with the command **nfisisom()** in PARI/GP).

Results

We applied the algorithm for every possible choice of $p(1) \pmod 2$, N and S_1, verifying 40 different cases.

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Theorem (B.)

Let d_K be the discriminant of a number field K with degree 8 and signature $(2,3)$. Then the minimum value of $|d_K|$ is equal to 4286875.

Theorem (B.)

There are 56 number fields of degree 8 and signature $(2,3)$ with $|d_K| \leq 5726300$; with the exception of two non-isomorphic fields with $|d_K| = 5365963$, every field in the list is uniquely characterized by the value of $|d_K|$.

$-d_K$	Factorization	$f(x)$
4286875	$5^4 \cdot 19$	$x^8 - 3x^7 - x^6 + 7x^5 + 3x^4 - 6x^3 - 4x^2 + x + 1$
4296211	$199 \cdot 21589$	$x^8 - x^7 + 3x^5 - 4x^4 + 2x^3 + 2x^2 - 3x + 1$
4297507	$2011 \cdot 2137$	$x^8 - 2x^6 - x^5 - x^3 + 2x^2 + x - 1$
4364587	$29 \cdot 150503$	$x^8 - 3x^6 - 3x^5 + 4x^4 + 7x^3 - 2x^2 - 4x - 1$
4386467	$41 \cdot 83 \cdot 1289$	$x^8 + 4x^6 - 2x^5 + 3x^4 - 5x^3 + x^2 - 2x + 1$
4421387	$1321 \cdot 3347$	$x^8 - x^6 - x^5 + 2x^4 - x^3 - 2x^2 + 2x - 1$
4423907	prime	$x^8 - 2x^5 - 5x^4 - 5x^3 - 5x^2 - 2x - 1$
4456891	prime	$x^8 - 3x^6 - 3x^5 + 5x^4 + 6x^3 - 2x^2 - 4x - 1$
4461875	$5^4 \cdot 11^2 \cdot 59$	$x^8 - x^7 + x^6 + 2x^5 - 2x^4 + 2x^2 - x - 1$
4505651	prime	$x^8 - 3x^6 - 3x^5 + 5x^4 + 4x^3 - 3x^2 - x + 1$
4542739	prime	$x^8 - 4x^6 - 3x^5 + 6x^4 + 7x^3 - x^2 - 4x - 1$
4570091	$1249 \cdot 3659$	$x^8 - x^6 - x^5 + x^4 - x^3 - 2x^2 + x + 1$
4570723	prime	$x^8 - 2x^7 + x^6 + 3x^5 - 5x^4 - 3x^3 + 4x^2 + x - 1$
4584491	$19 \cdot 101 \cdot 2389$	$x^8 - 3x^6 - x^5 + 3x^4 + 4x^3 - x^2 - 3x - 1$
4596992	$2^8 \cdot 17957$	$x^8 - 3x^6 - 2x^5 + 3x^4 - x^2 + 2x - 1$

$-d_K$	Factorization	$f(x)$
4603987	prime	$x^8 - x^7 - 4x^6 - 3x^5 + 3x^4 + 8x^3 + 8x^2 + 4x + 1$
4614499	prime	$x^8 - x^6 - 3x^5 + x^4 + 2x^3 - x^2 + x + 1$
4616192	$2^{12} \cdot 7^2 \cdot 23$	$x^8 - 2x^6 - 2x^5 + 2x^4 + 4x^3 + x^2 - 2x - 1$
4623371	$17 \cdot 31^2 \cdot 283$	$x^8 + x^6 - x^3 - x^2 - 1$
4648192	$2^8 \cdot 67 \cdot 271$	$x^8 - x^6 - 2x^5 - 2x^4 + 2x^2 + 2x + 1$
4663051	$31 \cdot 359 \cdot 419$	$x^8 - x^7 + x^6 - 3x^5 + 7x^4 - 6x^3 + x^2 + 2x - 1$
4690927	$443 \cdot 10589$	$x^8 - 4x^6 - 4x^5 + 3x^4 + 6x^3 - x^2 - 3x + 1$
4711123	$43 \cdot 331^2$	$x^8 + 2x^6 - 7x^5 - 4x^4 - 9x^3 + 9x^2 + 6x + 1$
4725251	$59 \cdot 283^2$	$x^8 - 4x^6 - 2x^5 + 7x^4 + 5x^3 - 3x^2 - 4x - 1$
4761667	$23 \cdot 207029$	$x^8 - 3x^6 - 2x^5 - 2x^4 + 3x^3 + 9x^2 + 6x + 1$
4775363	$1931 \cdot 2473$	$x^8 - 6x^6 - 2x^5 + 9x^4 + x^3 - 5x^2 + 1$
4785667	$29 \cdot 59 \cdot 2797$	$x^8 - x^5 - 4x^4 - 3x^3 + 2x^2 + 3x + 1$
4809907	$19 \cdot 253153$	$x^8 - 4x^6 - x^5 + 5x^4 + x^3 - x^2 - x - 1$
4858379	$17^2 \cdot 16811$	$x^8 + 3x^6 - x^5 + 2x^4 - 3x^3 - 2x + 1$
4931267	$11 \cdot 67 \cdot 6691$	$x^8 - x^6 - x^5 - 6x^4 - 2x^3 + 17x^2 - 8x + 1$
4960000	$2^8 \cdot 5^4 \cdot 31$	$x^8 - x^6 - 6x^5 + 6x^4 - 2x^3 + 8x^2 - 6x + 1$

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5040467	prime	$x^8 - 5x^6 - 3x^5 + 6x^4 + 4x^3 - 3x^2 - 2x + 1$
5040547	$37 \cdot 59 \cdot 2309$	$x^8 - 2x^6 + 3x^4 - 3x^3 - 3x^2 + 4x + 1$
5103467	prime	$x^8 - 5x^6 - x^5 + 8x^4 + 2x^3 - 4x^2 - x + 1$
5107019	prime	$x^8 - 3x^6 - 3x^5 + 3x^4 + 9x^3 + 6x^2 + x - 1$
5118587	$29 \cdot 176503$	$x^8 - 2x^6 - 5x^5 - 6x^4 + 11x^3 + 20x^2 + 9x + 1$
5149367	$47 \cdot 331^2$	$x^8 - 2x^6 - 2x^5 + 8x^4 - 2x^3 - 5x^2 + 4x - 1$
5155867	$449 \cdot 11483$	$x^8 - 3x^6 - x^5 + 3x^4 + x^3 - 2x^2 - x + 1$
5165819	$641 \cdot 8059$	$x^8 - 6x^6 - 5x^5 + 5x^4 + 9x^3 + 6x^2 + 2x + 1$
5204491	prime	$x^8 - 6x^6 - 7x^5 + 8x^4 + 19x^3 + 15x^2 + 6x + 1$
5233147	prime	$x^8 + 2x^6 - x^5 - 11x^4 - 9x^3 + 2x^2 + 4x + 1$
5272027	$317 \cdot 16631$	$x^8 + x^6 - 7x^5 + 6x^4 - 4x^3 + 5x^2 - 4x + 1$
5286727	prime	$x^8 - 4x^6 + 5x^4 - 3x^2 - x + 1$
5293867	$227 \cdot 23321$	$x^8 - 4x^6 - x^5 + 8x^4 + 5x^3 - 6x^2 - 5x + 1$
5344939	$521 \cdot 10259$	$x^8 - 5x^6 - 4x^5 + 5x^4 + 16x^3 + 5x^2 - 6x + 1$
5346947	$839 \cdot 6373$	$x^8 - 6x^6 - 3x^5 + 9x^4 + 7x^3 + x^2 + x + 1$
5359051	prime	$x^8 - 4x^6 - 3x^5 + 3x^4 + 11x^3 + 10x^2 + 4x + 1$

$-d_K$	Factorization	$f(x)$
5365963	$67 \cdot 283^2$	$x^8 - x^6 - 2x^5 + 5x^3 + 5x^2 + 4x + 1$
5365963	$67 \cdot 283^2$	$x^8 - 3x^5 - 5x^4 - 5x^3 + 11x^2 - x + 1$
5369375	$5^4 \cdot 11^2 \cdot 71$	$x^8 + 4x^6 - 6x^5 + 6x^4 - 12x^3 - 7x^2 - 6x + 1$
5371171	$13 \cdot 413167$	$x^8 - x^6 - 5x^5 + 2x^4 + 9x^2 - 6x + 1$
5420747	prime	$x^8 - 5x^6 - 4x^5 + 5x^4 + 8x^3 + 5x^2 + 2x + 1$
5525731	$17 \cdot 325043$	$x^8 - 5x^6 - 3x^5 + 3x^4 - 2x^3 - 8x^2 - 4x + 1$
5635607	$61 \cdot 92387$	$x^8 + 2x^6 - 5x^5 - 6x^4 + 8x^3 + 2x^2 - 4x + 1$
5671691	$193 \cdot 29387$	$x^8 - 3x^6 - 3x^5 + 4x^4 + 3x^3 - 2x^2 + 1$
5697179	prime	$x^8 + x^6 - 8x^4 - 3x^3 + 5x^2 + 2x + 1$

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However, they explicitly made no claim of complete classification.
- Actually, all the minimum polynomials were found in a previous attempt with $|d_K| \leq 5000000$. This suggests that this method is somehow too coarse.

Thank you for your attention.