Diophantine problems with prime variables

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This is a report on joint work with Alessandro Gambini and Alessandro Languasco [1], where we improve the results in [2]. We deal with a Diophantine approximation problem with prime variables: the goal is to prove that the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega| \le \left(\max\{p_1, p_2, p_3^k\}\right)^{-\psi(k)+\varepsilon} \tag{1}$$

has infinitely many solutions in prime variables p_1, p_2 and p_3 for any given real number ω, under as mild Diophantine assumptions on the real constants λ_1, λ_2 and λ_3 as possible and with $\psi = \psi(k) > 0$ in an interval $(1, k_0]$. In [2] we proved that we can take $\psi(k) = (4-3k)/(10k)$ for $k \in (1, 4/3)$. Now we improve on this result both in the admissible range for k and in the exponent $\psi(k)$ in (1), in the common range.

Assume that $1 < k \le 3$, λ_1, λ_2 and λ_3 are non-zero real numbers, not all of the same sign, that λ_1/λ_2 is irrational and let ω be a real number. The inequality (1) has infinitely many solutions in prime variables p_1, p_2, p_3 for any $\varepsilon > 0$, where

$$\psi(k) = \begin{cases} (3-2k)/(6k) & \text{if } 1 < k \le \frac{6}{5}, \\ 1/12 & \text{if } \frac{6}{5} < k \le 2, \\ (3-k)/(6k) & \text{if } 2 < k < 3, \\ 1/24 & \text{if } k = 3. \end{cases} \tag{2}$$

It is easily seen that the hypothesis on the sign is natural, if one wants to approximate all real numbers, whereas the hypothesis on the ratio λ_1/λ_2 is needed to avoid trivial cases when (1) can not hold. For the proof we use a modern variant of the technique introduced in the 1930's by Davenport & Heilbronn. The values for ψ given by (2) depend on suitable bounds for the relevant exponential sums over prime powers.

References

[1] A. Gambini, A. Languasco, and A. Zaccagnini, *A Diophantine approximation problem with two primes and one k-th power of a prime*, submitted. Arxiv preprint 1706.00343.

[2] A. Languasco and A. Zaccagnini, *A Diophantine problem with prime variables*, Highly Composite: Papers in Number Theory, Proceedings of the "International Meeting in Number Theory," celebrating the 60th birthday of Prof. R. Balasubramanian, Harish-Chandra Research Institute, Allahabad, Dec. 2011 (V. Kumar Murty, D. S. Ramana, and R. Thangadurai, eds.), Ramanujan Mathematical Society–Lecture Notes Series, vol. 23, 2016, pp. 157–168.

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