Structural Properties of Weakly Directed Families with Applications to Non-Unique Factorization Theory

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Let $\mathscr{L}$ be a collection of non-empty subsets of $\mathbf{N}$. Given $k \in \mathbf{N}$, we write $\mathscr{U}_k$ for the union of all $L \in \mathscr{L}$ with $k \in L$. We say that $\mathscr{L}$ satisfies:

• the Structure Theorem (for Unions) if there exist $d \in \mathbf{N}^+$ and $M \in \mathbf{N}$ such that, for all large $k \in \mathbf{N}$, $\mathscr{U}_k \subseteq k + d \cdot \mathbf{Z}$ and $\mathscr{U}_k \cap [\![\inf \mathscr{U}_k + M, \sup \mathscr{U}_k - M]\!]$ is an arithmetic progression (shortly, AP) with difference d.

• the Strong Structure Theorem (for Unions) if there are $m \in \mathbf{N}^+$ and finite sets $\mathscr{U}_0', \mathscr{U}_0'', \ldots, \mathscr{U}_{m-1}', \mathscr{U}_{m-1}'' \subseteq \mathbf{N}$ such that, for all but finitely many k,

$$\mathscr{U}_k = \left(\inf \mathscr{U}_k + \mathscr{U}'_{k \bmod m}\right) \uplus \mathscr{P}_k \uplus \left(\sup \mathscr{U}_k - \mathscr{U}''_{k \bmod m}\right) \subseteq k + \delta' \cdot \mathbf{Z},$$

where $\mathscr{P}_k$ is an AP with difference δ' (in particular, this implies that $\mathscr{L}$ satisfies the Structure Theorem).

On the other hand, we let $\mathscr{L}$ be a *weakly directed family* if, for all $L_1, L_2 \in \mathscr{L}$, there is $L \in \mathscr{L}$ containing the sumset $L_1 + L_2$.

I've recently proved in [3] that weakly directed families satisfy the [Strong] Structure Theorem under mild conditions that are often met in various contexts: My talk will be about these results and their application to the theory of non-unique factorization [2, 1], with an emphasis on the case of transfer Krull domains of finite type (such as the ring of integers of a number field).

References

[1] Y. Fan and S.T., *Power monoids: A bridge between Factorization Theory and Arithmetic Combinatorics* e-print, `arxiv.org/abs/1701.09152`.

[2] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics **278**, Chapman & Hall/CRC, 2006.

[3] S.T., *Periodic properties of weakly directed families with applications to factorization theory*, e-print (`arxiv.org/abs/1706.03525`).