Generalized Vandermonde Determinants and Characterization of Divisibility Sequences

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Let us consider a field $\mathbb{F}$ of zero characteristic. A sequence $(a_n)_{n=0}^{+\infty}$ in $\mathbb{F}$ is a *divisibility sequence* if $a_0 = 0$, $a_1 = 1$ and there exists a subring $\mathcal{A}$ of $\mathbb{F}$, of finite type over $\mathbb{Z}$, such that for all n and $d \geq 1$ we have $\frac{a_{nd}}{a_n} \in \mathcal{A}$. We consider linear recurrence sequences which also are divisibility sequences in the non–degenerate case, i.e., when the ratio of two distinct roots of their minimal polynomial is not a root of unity, and obviously all the roots are different from zero. We present a different proof of a theorem on the characterization of linear recurrence sequences, which also are divisibility sequences, due by Van der Poorten, Bzivin, and Pethö. They considered a non–degenerate linear recurrence sequence $(a_n)_{n=0}^{+\infty}$, whose characteristic polynomial has distinct roots. Using the Hadamard quotient theorem and the theory of exponential polynomials they stated that if such a sequence is a divisibility sequence, then there is a resultant sequence $(\bar{a}_n)_{n=0}^{+\infty}$ such that

$$\forall n \geq 0 \quad a_n \mid \bar{a}_n,$$

where $\bar{a}_n$ has the shape

$$\bar{a}_n = n^k \prod_i \left(\frac{\alpha_i^n - \beta_i^n}{\alpha_i - \beta_i} \right). \tag{1}$$

Our proof is based on an interesting determinant identity, obtained from the evaluation of a generalized Vandermonde determinant. As a consequence of this new elementary proof we can find a more precise form for the resultant sequence (1), in the general case of non–degenerate divisibility sequences having minimal polynomial with multiple roots.